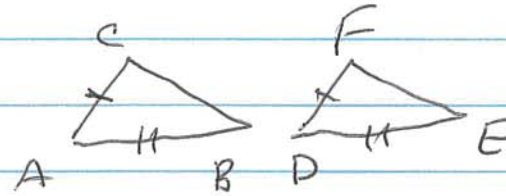
Rigid motion

Original idea to prove SAS, ASA, SSS

SAS



Move ΔDEF so $\angle FDE$ sits over $\angle CAB$

Notice D goes to A & E goes to B & F to C

since rigid motions preserve angle measures and distances.

To be logically rigorous, need a concept of motion, but none formally in Greek geom.

Fortunately?? can use linear algebra to justify this. Motion = Function $P \circlearrowleft$ or $S \circlearrowleft$

Theorem from Math 132

isometry $T: \mathbb{R}^2 \to \mathbb{R}^2$ 1-1 onto, distance pres.

$d(v,w) = d(Tv, Tw)$. Then there are real numbers a, b, c, d so that

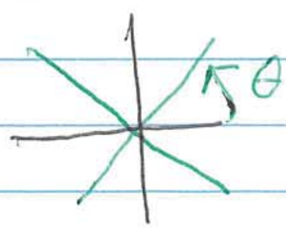
$a^2 + b^2 = 1$ and

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & \mp b \\ b & \pm a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} =$$

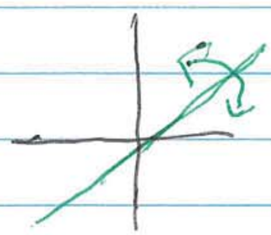
$$\begin{pmatrix} ax \mp by + c \\ bx \pm ay + d \end{pmatrix}$$

Special cases: $c = d = 0$ linear $\left\{\begin{array}{l}\text{mapping} \\ \text{transformation}\end{array}\right.$

det $= +1$ rotation through $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

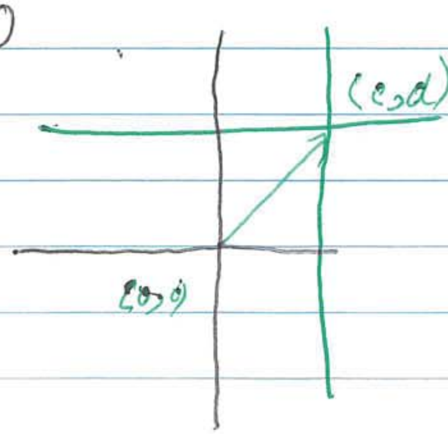


det $= -1$



reflection with respect to some line

$a = 1, b = 0$



translation.

Similarly in 3D with 3×3 matrix whose columns have unit length and are mutually perpendicular.

Converse: Every transformation of this type is an isometry. (Also Math 132)

Isometry: f 1-1 onto, $|f(X)f(Y)| = |XY|$.

Isometries and collinearity

First. $f: \mathbb{R}^2 \to \mathbb{R}^2$ isometry $\Rightarrow$ so is f^{-1}

$f, g: \mathbb{R}^2 \to \mathbb{R}^2$ isometries $\Rightarrow$ so is $g \circ f$.

Thm. (i) f isometry + $A * B * C \Rightarrow$ $f(A) * f(B) * f(C)$ and conversely (look at f^{-1}).

(ii) f isometry and $\{A, B, C\}$ collinear $\Rightarrow \{f(A), f(B), f(C)\}$ collinear, and conversely

Proof. (i) $A * B * C \Rightarrow |AC| = |AB| + |AC|$

L.H.S. $= |f(A)f(C)|$ R.H.S. $= |f(A)f(B)| + |f(B)f(C)|$.

Hence $|f(A)f(C)| = |f(A)f(B)| + |f(B)f(C)|$, so $f(A) * f(B) * f(C)$. The converse holds by going backwards and using the fact that f^{-1} is an isometry.

(*ii*) A, B, C collinear $\Rightarrow$ one is between other two. By (i) $A*B*C$ $\Rightarrow f(A)*f(B)*f(C)$, so $\{f(A), f(B), f(C)\}$ collinear. The cases $B*C*A$ & $C*A*B$ follow by interchanging roles of A, B, C. — Conversely if $\{f(A), f(B), f(C)\}$ collinear then one between other two. By (i) $f(A)*f(B)*f(C) \Rightarrow A*B*C$ as in preceding argument. The cases $f(B)*f(C)*f(A)$ and $f(C)*f(A)*f(B)$ follow similarly. ∎

Cor. $\{A, B, C\}$ noncollinear $\Leftrightarrow \{f(A), f(B), f(C)\}$ noncollinear. **Insert page 4A here.**

Note In 3D, isometries send coplanar sets to coplanar sets and noncoplanar sets to noncoplanar sets. — The most efficient way to do this is using linear algebra.

<u>Similarities</u> $|f(X)f(Y)| = r\,|XY|$, some $r > 0$ (same r for all point pairs).

<u>Theorem</u> All similarity transformations $T: \mathbb{R} \to \mathbb{R}$ have the form $c\,T_0$ where $c > 0$ and T_0 is an isometry.
(Also 2nd linear algebra course.)

<u>Consequences</u> T similarity

(i) $A*B*C \Longleftrightarrow T(A)*T(B)*T(C)$

(ii) T sends collinear points to collinear points and noncollinear points to noncollinear points.