Math 133
Fall 2021

MORE SOLUTIONS FOR WEEK 06 EXERCISES

For these exercises assume that the coordinate plane satisfies the axioms for Euclidean plane geometry, with distances and lines as described in Chapter 17 of Moise. Most of what we need about rigid motions and similarities appears in `lecture12.pdf`.

Definition. If F_1 and F_2 are subsets of a neutral geometry, then a 1–1 onto mapping $h : F_1 \to F_2$ is called an *isometry* (or a *congruence*) if for all $X, Y \in F_1$ we have $h(X)h(Y)| = |XY|$. — All of the exercises below are valid in a neutral geometry.

0. The main step in showing an isometry h preserves collinear and noncollinear points is the observation that for all distinct points X, Y, Z in the domain, we have $X * Y * Z$ holds if and only if $h(X)*h(Y)*h(Z)$; this in turn follows from the fact that $U*V*W$ if and only if $|UW| = |UV|+|VW|$.

To see that the inverse of an isometry is an isometry, let h be an isometry; we need to show that $|h^{-1}(X)h^{-1}(Y)| = |XY|$ for all X and Y. If we write $X = h(U)$ and $Y = h(V)$, for suitable U and V, then $U = h^{-1}(X)$ and $V = h^{-1}(Y)$ imply that

$$|h^{-1}(X)h^{-1}(Y)| \quad = \quad |UV| \quad = \quad |h(U)h(V)| \quad = \quad |XY|$$

where the second step is valid because h is an isometry. Therefore the inverse map h^{-1} is also an isometry.

Finally, if h and k are composable isometries, then

$$|k\circ h(X)k\circ h(Y)| \quad = \quad |h(X)h(X)| \quad = \quad |XY|$$

and therefore $k\circ h$ is also an isometry.■

1. Let $M \subset F_1$ be a maximal collinear subset. Since isometries take collinear sets to collinear sets, the image $h[M]$ is collinear. Let $N \supset h[M]$ be a maximal collinear subset containing $h[M]$. Then $h^{-1}[N]$ is a collinear subset containing M, and by maximality we must have $M = h^{-1}[N]$. Since h is 1–1 and onto, it follows that $N = h[M]$.■

2. (a) The maximal collinear subsets of $\angle ABC$ and $\angle DEF$ are $[BA$ and $[BC$ for $\angle ABC$ and Explain why h maps $[BA$ to either $[ED$ or $[EF$ and $[ED$ and $[EF$ for $\angle DEF$, and therefore the statements about the images of $[AB$ and $[BC$ follow from the preceding exercise. Since B is the unique point in the intersection of the maximal collear subsets $[BA$ and $[BC$ and E is the unique point in the intersection of the maximal collear subsets, the preceding sentence and the set-theoretic identity $h\,[\mathcal{U}\cap\mathcal{V}] \quad = \quad h\,[\mathcal{U}]\cap h\,[\mathcal{V}]$, which holds since h is 1–1 onto, now imply that $h(B) = E$.■

(b) Without loss of generality we may assume that the isometry h sends $[BA$ and $[BC$ to $[DE$ and $[DF$ respectively; in the other case where h sends $[BA$ and $[BC$ to $[DF$ and $[DE$ respectively one can Let $D' \in (ED$ and $F' \in (EF$ be such that $|ED'| = |BA|$ and $|EF'| = |BC|$. Since $D' \in (ED$ and $F' \in (EF$ are uniquely determined by the distance condition, it follows that $h(A) = E'$ and $h(C) = F'$.

The preceding imply that $\triangle ABC \cong \triangle D'EF'$ by SSS, so that $|\angle ABC| = |\angle D'EF'|$. Since $\angle D'EF' = \angle DEF$ by construction, it follows that $|\angle ABC| = |\angle DEF|$.■

3. (a) The crucial step is to describe the maximal collinear subsets of the square. Clearly the edges $[AB]$, $[BC]$, $[CD]$ and $[AD]$ are all collinear sets, and we claim each is maximal; without loss of generality we may restrict attention to $[AB]$, for the other cases will then follow by interchanging the roles of the vertices. If $[AB]$ were contained in a larger collinear subset which also includes the point X, there are two cases depending upon whether or not $X \in [BC] \cup [AD]$. — If the former holds, then $X \notin [AB]$ implies that $X \notin AB$ because if it were then both AB and either AD or BC would contain the subset $\{A, X\}$ or $\{B, X\}$ respectively, and we know this is not the case. Finally, since CD is parallel to AB (both are perpendicular to each of AD and BC), it follows that $[CD]$ and AB are disjoint.

By the preceding discussion, each vertex is the unique point in an intersection of two maximal collinear subsets, and every non-vertex point lies on exactly one maximal collinear subset. By the reasoning in the previous exercise, it follows that the isometry must send vertices to vertices and non-vertices to non-vertices. Therefore h must send $\{A, B, C, D\}$ into itself.

Assume now that $h(A) = A$. Then h sends each of the maximal collinear subsets containing A to a subset of the same type. These subsets are $[AB]$ and $[AD]$, so either h sends each subset into itself or else it switches them. Since h sends vertices to vertices, it follows that either $h(B) = D$ and $h(D) = B$ or else h sends these vertices to themselves. In either case we are only left with one possibility for $h(C)$; namely, C itself. The final sentence follows if we replace A by one of B, C, D in the preceding discussion.$\blacksquare$

(b) As before, it will suffice to prove that $h(X) = X$ for all $X \in [AB]$; the other cases will follow by permuting the roles of the vertices. If $X \in [AB]$, we know that $h(X) \in [AB]$ by part (a) and the fact that isometries preserve betweenness. Furthermore, we have $|A\,h(X)| = |AX|$ since h is an isometry and $h(A) = A$. For each $d \geq 0$ there is a unique point Y on $[AB$ such that $|AY| = d$ by the Ruler Postulate, andsince $h(X) \in [AB$ it follows that $h(X)$ must be equal to X.$\blacksquare$

4. (a) We are given that $|XY| \leq M$ for all $X, Y \in F_1$ and there are points $X_0, Y_0 \in F_1$ such that $|X_0Y_0| = M$. Therefore $|h(X)h(Y)| \leq M$ for all $X, Y \in F_1$ and there are points $X_0, Y_0 \in F_1$ such that $|h(X_0)h(Y_0)| = M$. Since h is onto, the first statement implies that $|UV| \leq M$ for all $U, V \in F_2$, and the second statement implies that $|U_0V_0| = M$ for some $U_0, V_0 \in F_2$. Therefore F_2 also has a proper diameter equal to M.$\blacksquare$

(b) If we have $a \leq s \leq t \leq b$, then $t - s \leq b - a$ and therefore $[AB]$ has a proper diameter equal to $b - a$; similarly, $[CD]$ has a proper diameter equal to $d - c$. Therefore if $f : [a, b] \to [c, d]$ is an isometry these two diameters must be equal.$\blacksquare$

(c) It will suffice to show that (AB) does not have a proper diameter. We know that $|XY| \leq b - a$ if $X, Y \in (AB)$, but now $a < s \leq t < b$ implies that $t - s < b - a$. If there were a proper diameter $M = t_0 - s_0$ and we let $t_1 = \frac{1}{2}(y_+ b)$, then we would have $t_1 - s_0 > M$; this contradicts the assumption that there is some proper diameter M and therefore none can exist.$\blacksquare$

(d) Let r denote the radius of a circle Γ. The basic results on circles state that if $X, Y \in \Gamma$, then $|XY| \leq 2r$ and $|XY| = 2r$ if $[XY]$ is a diameter, so that Γ has a proper diameter equal to $2r$. Now assume that Γ_1 and Γ_2 are circles whose radii are equal to a and b respectively and that $h : \Gamma_1 \to \Gamma_2$ is an isometry. By (a) their proper diameters must be equal, so that $2a = 2b$, and thus it follows that $a = b$.$\blacksquare$