Introduction to non – Euclidean geometry

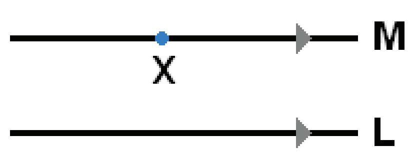
Mathematics 133, Fall 2021

> Over the course of the nineteenth century, under pressure of developments within mathematics itself, the accepted answer [to questions like, "What is geometry**?**"] dramatically broke down. … Not since the ancient Greeks, if then, had there been such an irruption [or incursion] of philosophical ideas into the very heart of mathematics. … Mathematicians of the first rank … found themselves obliged to confront questions about … the status of geometry … The answers they gave did much to shape the mathematics of the twentieth century.
>
> W. Ewald (1954 –), ***Bulletin (New Series) of the American Mathematical Society***, Vol. **40** (2002), pp. 125 – 126.

The fifth and final postulate in Euclid's ***Elements*** differs from the latter's other assumptions in several respects. All of the remaining statements are fairly simple (for example, lines can be extended indefinitely in either direction, or a circle can be drawn with arbitrary center and radius), but the last one is fairly complicated by comparison. In particular, it takes more words to state this postulate (both in English and the original Greek) than are needed for the remaining four postulates combined. Furthermore, the proofs of the first 28 results in the ***Elements*** do not use the Fifth Postulate. In addition, there are general questions whether this postulate corresponds to physical reality because it involves objects which are too distant to be observed or questions about measurements that cannot necessarily be answered conclusively because there are always limits to the precision of physical measurements.

For these and other reasons, it is natural to speculate about the extent to which the exceptional Fifth Postulate is necessary or desirable as an assumption in classical geometry. Historical evidence suggests that such questions had been raised and debated extensively before Euclid's time, and for centuries numerous mathematicians tried to prove the Fifth Postulate from the others, or at least to find a simpler and more strongly intuitive postulate to replace it. Progress on the second issue is reflected by the following equivalent assumption, which was suggested in the 5th century A. D. by Proclus Diadochus (410 – 485) and subsequently by J. Playfair (1748 – 1819):

"Playfair's Postulate": *Given a line* **L** *and a point* **X** *which is not on* **L**, *there is a unique line* **M** *such that* $\mathbf{X} \in \mathbf{M}$ *and* **L** *is parallel to* **M**.



During the 18th century several mathematicians made sustained efforts to understand what would happen if the Fifth Postulate (and logically equivalent statements) were false, and in early 19th century a few mathematicians concluded that such efforts would not yield a logical contradiction, and they concluded there was a logically sound alternative to the truth of the Fifth Postulate. Later in that century other mathematicians proved results vindicating this conclusion; in particular, such results prove the logical impossibility of proving the Fifth Postulate or replacing it by something that raises fewer questions.

The discovery of non – Euclidean geometry had major implications for the role of geometry in mathematics, the sciences and even philosophy. The following three quotations summarize this change as it evolved from the 17th century through the beginning of the 20th century.

> Geometry is the basic mathematical science, for it includes arithmetic, and mathematical numbers are simply the signs of geometrical magnitude.
>
> Isaac Barrow (1630 – 1677), ***Mathematical Lectures*** (1664 – 1666).
>
> The concept of [Euclidean] space is by no means of empirical origin, but is an inevitable necessity of thought.
>
> Immanuel Kant (1724 – 1804), ***Critique of Pure Reason*** (1781).
>
> I am convinced more and more that the necessary truth of our geometry cannot be demonstrated, at least not ***by*** the ***human*** intellect ***to*** the human understanding. Perhaps in another world, we may gain other insights into the nature of space which at present are unattainable to us. Until then we must consider geometry as of equal rank not with arithmetic, which is purely ***a priori***, but with mechanics.
>
> C. F. Gauss (1777 – 1855), ***Letter to H. W. M. Olbers*** (1817). [***Note:*** Olbers (1758 – 1840) was an astronomer, physician and physicist, and he is known as the discoverer of the asteroid Pallas.]
>
> One geometry cannot be more valid than another; it can only be more convenient.
>
> Henri Poincaré (1854 – 1912), ***Science and Hypothesis*** (1901).
>
> We have Einstein's space, De Sitter's space, expanding universes, contracting universes, vibrating universes, mysterious universes. In fact, the pure mathematician [or theoretical physicist] may create universes just by writing down an equation, and indeed if he is an individualist he can have a universe of his own.
>
> J. J. Thomson (1856 – 1940) [discoverer of the electron]

In these notes we shall discuss the mathematical theory (in fact, multiple theories) obtained by not assuming the Fifth Postulate, and we shall also include further comments on the role of geometry in modern mathematics and science.

The treatment of non – Euclidean geometry in Moise is different from the one presented here, and the reason for our alternate treatment is that it is closer in spirit to the material on Euclidean geometry. Moise's approach involves more sophisticated methods which yields stronger conclusions (these are noted in Section 4). The details appear in Chapters 9, 10 and 24.

1 : Facts from spherical geometry

The sphere's perfect form has fascinated the minds of men for millennia.
From planets to raindrops, nature … [makes use of] the sphere.

http://www.bbc.co.uk/radio4/science/fiveshapes.shtml

Spherical geometry can be said to be the first non – Euclidean geometry.

D. Henderson (1939 – 2018) and D. Taimina (1954 –), ***Math. Assoc. of America Notes*** No. **68** (2005), p. 59.

Before we discuss the material generally known as non – Euclidean geometry, it will be helpful to summarize a few basic results from spherical geometry.

As noted in the following quote from **http://www.physics.csbsju.edu/astro/CS/CSintro.html**, it is natural to think of the sky as a large spherical dome and to use this as a basis for describing the positions of the stars and other heavenly bodies:

> If you go out in an open field on a clear night and look at the sky, you have no indication of the distance to the objects you see. A particular bright dot may be an airplane a few miles off, a satellite a few hundred miles off, a planet a many millions of miles away, or a star more than a million times further away than the most distant planet. Since you can only tell direction (and not distance) you can imagine that the stars that you see are attached to the inside of a spherical shell that surrounds the Earth. The ancient Greeks actually believed such a shell really existed, but for us it is just a convenient way of talking about the sky.

The following photograph of a planetarium display provides a graphic illustration:

In fact, the historical relationship between astronomy and spherical geometry goes much further than simple observations. When some of the first attempts at scientific theories of astronomy were developed by Eudoxus of Cnidus (408 – 355 B.C.E.) and Aristotle (384 – 322 B.C.E.) during the 4th century B.C.E., the ties between spherical geometry and astronomy became even closer, and both of these subjects were studied at length by later Greek scientists and mathematicians, including Hipparchus of Rhodes (190 – 120 B.C.E.), Heron of Alexandria (*c.* 10 A.D. – 75), Menelaus of Alexandria (70 – 130), and (last but not least) Claudius Ptolemy (85 – 165).

Another subject which began to emerge at the same time was trigonometry, which was studied both on the plane and on the surface of a sphere; not surprisingly, spherical trigonometry played a major role in efforts by ancient astronomers to explain the motions of stars and planets. In particular, during the Middle Ages both Arab and Indian mathematicians advanced spherical trigonometry far beyond the work of the ancient Greek mathematicians. During the later Middle Ages, practical questions about navigation began to influence the development of spherical geometry and trigonometry, and there was renewed interest which led to major advances in the subject continuing through the 18th century. Due to their importance for navigation and astronomy, spherical geometry and trigonometry were basic topics in high school mathematics curricula until the middle of the 20th century, but this has changed for several reasons (for example, one can use satellites and computers to do the work that previously required human vision and computation, and to do so more reliably). Our main purpose here is to describe the main aspects of spherical geometry, so some proofs and definitions will only be informal and other arguments will not be given at all.

Great circles

In Euclidean plane and solid geometry, one reason for the importance of lines is that they describe the shortest paths between two points. On the surface of a sphere, the shortest curves between two points are given by ***great circles***.

Definitions. Given a point **X** in $\mathbb{R}^3$ and $k > 0$, the ***sphere*** Σ ***of radius*** k and center **X** is the set of all points **Y** in $\mathbb{R}^3$ such that $|\mathbf{XY}| = k$. A ***great circle*** on this sphere Σ is a circle given by the intersection of Σ with a plane containing **X.**

The following result yields a large number of geometrically significant great circles.

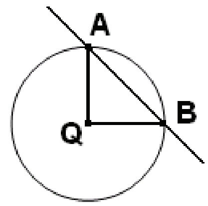
Proposition 1. *Let* Σ *be a sphere as above with center* **X**, *and let* **Y** *and* **Z** *be two points of* Σ.

1. *If* **X**, **Y** *and* **Z** *are not collinear, then there is a unique great circle on* Σ *containing* **Y** *and* **Z**.
2. *If* **X**, **Y** *and* **Z** *are collinear, then there are infinitely many great circles on* Σ *containing* **Y** *and* **Z**.

The second possibility arises when the segment **[YZ]** is a diameter of the sphere; in this case we often say that **Y** and **Z** are ***antipodal*** (pronounced ann–TIP–o–dal).

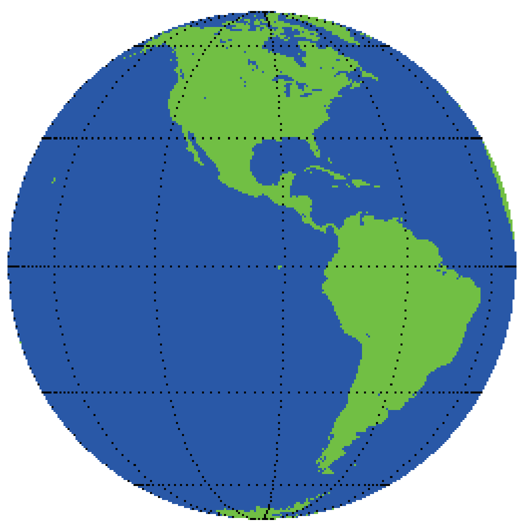
Sketch of proof. The results follow by considering set of all planes containing the three points. In the first case there is only one, but in the second there are infinitely many planes containing the line **YZ.** ■

Major and minor arcs. If **A** and **B** are points on a circle Γ, then they determine two arcs. If the points are antipodal, the two arcs are semicircles, and if they are not antipodal one has a ***major arc*** and a ***minor arc***. One way of distinguishing between these arcs is that the minor arc consists of **A, B** and all points on the circle Γ which lie on the ***opposite side*** of **AB** as the center **Q**, while the major arc consists of **A, B** and all points on the circle Γ which lie on the ***same side*** of **AB** as **Q.** We shall denote the minor arc by the symbol **€(AB)** (*Euro sign*).



With this terminology we can describe the shortest curve(s) joining two points **A** and **B** on the sphere more precisely as follows: If the points are antipodal, the shortest curve is any semicircular arc joining **A** and **B** (such an arc is automatically a great circle), and if the points are not antipodal, it is the minor arc on the great circle determined by the points **A** and **B.** Actually proving these statements turns out to be a nonelementary exercise, and we shall not discuss the details here.

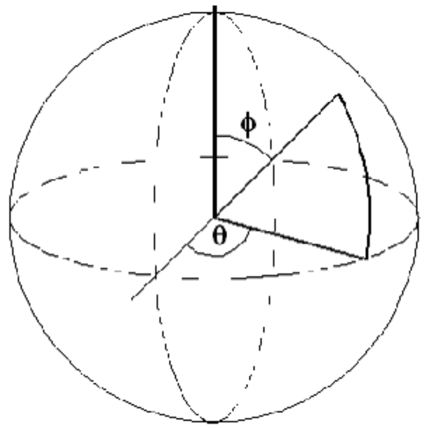
Latitude, longitude and spherical coordinates. The standard method for locating points on the surface of the earth is by means of latitude and longitude coordinates.



In fact, these are equivalent to the spherical coordinates that are used in multivariable calculus. Specifically, we specialize spherical coordinates to the sphere of radius a, the conversion from rectangular to spherical coordinates is given by

$$(x, y, z) \quad = \quad (a\cos\theta\sin\phi,\ a\sin\theta\sin\phi,\ a\cos\phi)$$

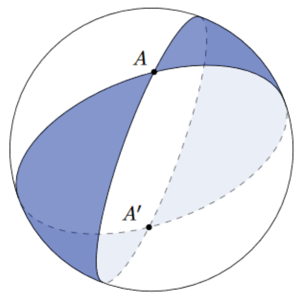
Then θ corresponds to the longitude counterclockwise from the meridian semicircle through $(0, 0, 1)$, $(0, 1, 0)$ and $(0, 0, -1)$, and ϕ corresponds to rescaled latitude, where 0 degrees represents the north pole and 180 degrees represents the south pole.



Lunes

Straight lines provide the basic pieces with which constructs familiar plane figures, and similarly great circle arcs provide the basic pieces for constructing spherical figures. Some of these are analogous to figures in the plane, but we shall start with one class of figures that is different.

In the plane, there are no interesting polygons with only two sides. This is not true on the sphere. A pair of great circles meets in two antipodal points, and these curves divide the sphere into four regions, each of which has two edges which are semicircles of great circles. The two semicircles bounding such a region form a ***lune*** (pronounced "loon"), or a ***biangle***; the first name reflects the fact that the regions bounded by lunes correspond to the phases of the moon that are visible from the earth at any given time.



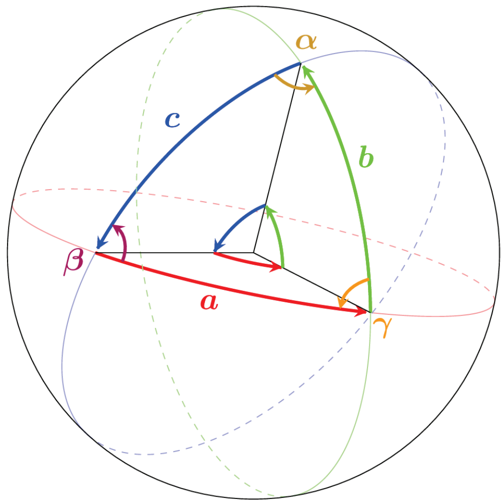
Lunes are fairly simple objects, but they have a few properties that we shall note:

1. The vertices of a lune are antipodal points.
2. The two vertex angles of a lune have equal measures.
3. The areas of the smaller and larger regions bounded by a lune are determined by the measures of these vertex angles. *

For the sake of completeness, we should note that the angles are measured using the tangent rays to the semicircles at the two vertex points where the latter meet.

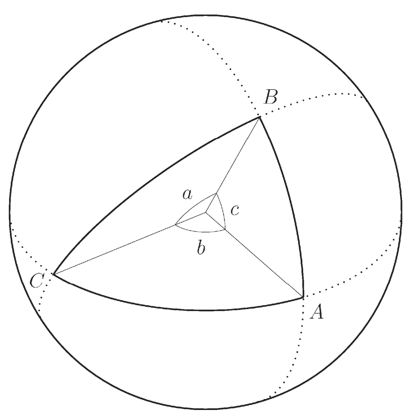
*** Area $= 2r\alpha$ where α is the measure of the vertex angle at A in radians and r is the sphere's radius.**

Spherical triangles are is defined just like planar triangles; they consists of three points which do not lie on a great circle and are called ***vertices***, and three arcs of great circles that join the vertices, which are called the ***sides***. An illustration is given below.



To simplify matters, we shall concentrate on <u>*small triangles*</u>, in which the sides are minor great circle arcs. Most if not all results actually hold for "large" triangles; the derivations are not particularly difficult, but here we are interested in describing the main points of spherical geometry rather than stating and proving the best possible results.

Just as there are six basic measurements associated to a plane triangle, there are also six basic measurements associated to a spherical triangle. In the picture below, they correspond to the ***degree measures of the minor great circle arcs*** joining **A** to **B**, **B** to **C**, and **A** to **C** (analogous to the lengths of the sides), and the ***measures of the vertex angles*** at **A**, **B** and **C**. These vertex angles are measured exactly like the vertex angles for lunes. For example, the vertex angle at **A** is measured by considering the lune formed by the two great semicircles which have **A** as one endpoint and pass through the points **B** and **C**. Of course, similar considerations apply to the other two vertex angles.



The basic geometry and trigonometry of spherical triangles has been worked out fairly completely, and in many respects it resembles the theory of ordinary plane triangles. In particular, one has analogs for the following theorems in plane geometry and trigonometry:

1. The Pythagorean Theorem (but the spherical formula is different!).
2. The **S.A.S., A.S.A., A.A.S.,** and **S.S.S.** congruence theorems for triangles.
3. The Law of Sines and the Law of Cosines.
4. Standard inequalities involving measurements of triangle parts (*e.g.*, the longer side is opposite the larger angle, the strict Triangle Inequality).

One major difference between plane and spherical triangles is that we have an A.A.A. congruence theorem for the latter.

<u>**Theorem 2.** (**A.A.A.** Congruence Theorem)</u> *Suppose that we are given two (small) spherical triangles on the same sphere* Σ *with vertices* **A, B, C** *and* **D, E, F.** *If the measures of the vertex angles at* **A, B, C** *are equal to the measures of the vertex angles at* **D, E, F** *respectively, then the lengths of the arcs* **€(AB), €(BC),** *and* **€(AC)** *are equal to the lengths of the arcs* **€(DE), €(EF),** *and* **€(DF)** *respectively.* ■

It is natural to ask why there is such a result for spherical triangles when the analog for plane triangles is completely false, and there is a fairly simple conceptual answer to this question which involves an important relationship between the surface area of the spherical region bounded by a spherical triangle and the sum of the measures of its vertex angles.

Many classic solid geometry textbooks from the first half of the 20th century contain proofs of these results. ■

Angle sums and surface area in spherical geometry

It is intuitively clear that a small spherical triangle with vertices **A, B,** and **C** bounds a closed region in the sphere which is analogous to the closed interior of a plane triangle; in particular, this can be seen from the two previous drawings of spherical triangles. Specifically, the closed "interior" region on the sphere determined by the spherical triangle is the union of the spherical triangle with the spherical region defined by the intersection of the following sets:

1. The sphere itself.
2. The set of all point in space on the same side of the plane **OAB** as **C.**
3. The set of all point in space on the same side of the plane **OAC** as **B.**
4. The set of all point in space on the same side of the plane **OBC** as **A.**

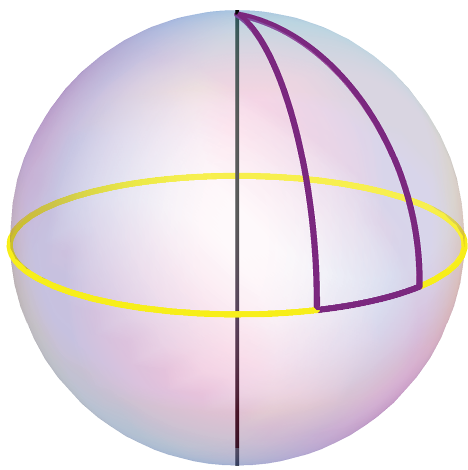
<u>Problem:</u> *What is the surface area of this closed region?*

The answer is given by the following result due to A. Girard (1595 – 1632).

Theorem 3. Let **A, B, C** *be the vertices of a spherical triangle as above, and assume that the sphere containing them has radius* k. *Let* α, β, γ *be the measures of the vertex angles of the spherical triangle above at* **A, B** *and* **C,** *all expressed in radians. Then the angle sum* $\alpha + \beta + \gamma$ *is greater than* π, *and the area of the closed region bounded by the spherical triangle with these vertices is equal to* $k^2(\alpha + \beta + \gamma - \pi)$.

Notation. The difference $\alpha + \beta + \gamma - \pi$ is called the ***spherical excess*** of the spherical triangle.

Here is an **example** to illustrate the conclusions: Consider the spherical triangle below, which has one vertex at the North Pole and two on the Equator. The measures of the angles at the equatorial vertices are both 90 degrees, and clearly we can take the measure E of the angle at the polar vertex to be anything between $90°$ and $180°$. The spherical excess of such a triangle (measured in degrees) is then equal to E, and the area of the spherical triangle is equal to $k^2\pi E/180$.

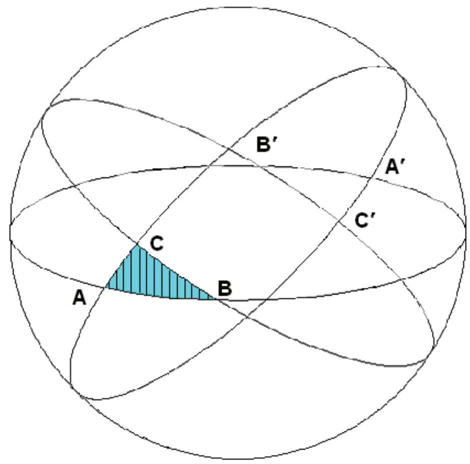


Incidentally, Heron's Formula for the area of a plane triangle in terms of the lengths of the sides has an analog for spherical triangles which is due to S. L'Huillier (1750 – 1840):

$$\tan\left(\tfrac{1}{4}E\right) = \sqrt{\tan\left(\tfrac{1}{2}s\right)\tan\left[\tfrac{1}{2}(s-a)\right]\tan\left[\tfrac{1}{2}(s-b)\right]\tan\left[\tfrac{1}{2}(s-c)\right]}.$$

In this formula E denotes the spherical excess of the triangle, while a, b, c represent the lengths of the sides opposite vertices **A, B, C** and $s = ½(a + b + c)$.

Sketch of a proof for Girard's Theorem. Consider the special case in which the spherical triangle lies on a closed hemisphere (for example, all points on the equator or the northern hemisphere); as in previous discussions, it is possible to retrieve the general case from such special cases. If the spherical triangle does lie on a closed hemisphere, then the great circles containing the arcs **€(AB), €(BC), €(AC)** split the sphere into eight closed regions as illustrated below:



We know the areas of all the lunes determined by the spherical triangle, and these areas also turn out to be equal to the sums of the areas of the pieces into which these lunes are cut by the various great circles. Algebraic manipulation of these identities yields the area formula stated in the theorem. Further details are given on pages 4 – 5 of the following online site:

https://www.math.purdue.edu/~arapura/460/spherical.pdf

Girard’s Theorem foreshadowed one of the most important results in non – Euclidean geometry that will be discussed in Section 4 of these notes. ■

2 : Attempts to prove Euclid's Fifth Postulate

We have already noted that questions about Euclid's Fifth Postulate are almost certainly at least as old as the ***Elements*** itself. Apparently the first known attempt to prove this assumption from the others was due to Posidonius (135 – 51 B. C. E.), and the question was discussed at some length in the writings of Proclus from the 5th century. A thorough description of all known attempts to answer this question is beyond the scope of these notes, but we shall note that the writings of Omar Khayyam (Ghiyās od – Dīn Abul – Fatah Omār ibn Ibrāhīm Khayyām Nishābūrī, 1048 – 1122) and Nasireddin/Naṣīr al – Dīn al – Ṭūsī (Muḥammad ibn Muḥammad ibn al – Ḥasan al – Ṭūsī, 1201 – 1274) anticipated some important aspects of the subject, and later work of J. Wallis (1616 – 1703) was also significant in several respects. None of these scholars succeeded in proving the Fifth Postulate, but in many cases they showed that it is logically equivalent to certain other statements that often seem extremely reasonable. For example, in the work of Proclus, the Fifth Postulate is shown to be equivalent to an assumption that ***the distance between two given parallel lines is bounded from above by some constant***, and Wallis showed that the Fifth Postulate is true if ***one can construct triangles that are similar but not congruent to a given one***.

New viewpoints and increasing sophistication

By the end of the 16th century, mathematics had begun to evolve well beyond the classical work of the Greeks and non – European cultures in the Middle East, India and China. This growth accelerated during the 17th century, which is particularly noteworthy for the emergence of coordinate geometry and calculus. Both of these had major implications for geometry. First of all, they answered many difficult problems of classical geometry in a fairly direct fashion. Furthermore, they led to new classes of problems that could be studied effectively and powerful new techniques, most notably through the use of rectangular coordinate systems. During the 18th century mathematics continued to expand in many different directions. In particular, mathematicians such as L. Euler (1707 – 1783) made many striking discoveries about Euclidean geometry that were (apparently) unknown to the Greeks, and in view of the increasing mathematical sophistication of the time it is not surprising that increasingly sophisticated efforts to prove Euclid's Fifth Postulate began to appear. In many cases, the basic idea was to assume this assumption is false and to obtain a contradiction; if this could be done, then one could conclude that the Fifth Postulate was a logical consequence of the other assumptions.

What if the Fifth Postulate is false?

> Though this be madness, yet there is method in it.
>
> Shakespeare, ***Hamlet***, Act 2, Scene 2, line 206

Sustained and extensive efforts by mathematicians to prove the Fifth Postulate began to emerge near the end of the 17th century. One of the earliest attempts was due to G. Saccheri (1667 – 1733). His work is particularly noteworthy in his approach; namely, his idea was to show that a contradiction results if one assumes the Fifth Postulate is false, and he went quite far in analyzing what would happen if this were the case. The results indicated that there were two distinct options if one did not necessarily assume the Fifth Postulate; one of them is Euclidean geometry and the other is a system which is like Euclidean geometry in many respects but also has some properties which seem bizarre at first glance. His conclusions were very accurate until the very end, where he dismissed the non – Euclidean alternative as "repugnant to the nature of a straight line." Saccheri's work was not widely known during the 18th century, and A. – M. Legendre (1752 – 1833) independently obtained many of his results as well as some others.

A few 18th century mathematicians drew conclusions that anticipated the breakthroughs of the next century. The 1763 dissertation of G. S. Klügel (1739 – 1812) pointed out mistakes in 28 purported proofs of the Fifth Postulate, and the author expressed doubt that any proof at all was possible. Perhaps the most prescient insights during this period were due to J. H. Lambert (1728 – 1777). He did not claim to prove the Fifth Postulate, but instead he speculated that the geometry obtained by assuming the negation of Fifth Postulate was the geometry of "a sphere of imaginary radius" (***i.e.***, the square of the radius is a negative real number). This probably seemed very strange to many of his contemporaries, but the advances of the 19th century show it reflects some very important aspects of non – Euclidean geometry.

Lambert's insights were taken further by F. K. Schweikart (1780 – 1859) and F. A. Taurinus (1794 – 1874). Schweikart developed the alternative explicitly as a subject in its own right and called it ***astral geometry***, speculating that it might be true in "the space of the stars." Taurinus proceeded to derive the formulas of the analytic geometry for the alternative system. These formulas are exactly what one obtains by taking the standard formulas from spherical geometry and trigonometry by substituting an imaginary number for the radius of the sphere; this provided a strong confirmation of Lambert's earlier speculation. Independently, Gauss had discovered the same relationships and become convinced that no mathematical proof of the Fifth Postulate from the other assumptions was possible.

Statements equivalent to the Fifth Postulate

We have noted that much of the work on the Fifth Postulate can be viewed as showing that various statements are logically equivalent to that postulate. Here is a long but not exhaustive list of theorems in Euclidean geometry that are logically equivalent to the Fifth Postulate.

1. If two lines are parallel to a third line, then they are parallel to each other.
2. The angle sum of a triangle is 180 degrees.
3. The angle sum of at least one triangle is 180 degrees.
4. There exists at least one rectangle.
5. There exist two parallel lines that are everywhere equidistant.
6. The distance between two parallel lines is bounded from below by a positive constant.
7. The distance between two parallel lines is bounded from above by a positive constant.
8. If a line meets one of two parallel lines, it meets the other.
9. There exist two similar but noncongruent triangles.
10. The opposite sides of a parallelogram have equal length.
11. Every line containing a point in the interior of an angle must meet at least one ray of the angle.
12. Through a given point in the interior of an angle there is a line that meets both rays of the angle.
13. Given any area function for the closed interiors of triangles, there exist triangles with arbitrarily large areas.
14. If two parallels are cut by a transversal, the alternate interior angles have equal measures.
15. The line joining the midpoints of two sides of a triangle is parallel to the third side and equal to half its length.
16. Given three noncollinear points, there is a circle containing all three of them.
17. The ratio of the circumference of a circle to its diameter is constant.
18. An angle inscribed in a semicircle is a right angle.
19. The Pythagorean Theorem.
20. Given two positive real numbers a and b, there is a rectangle whose sides have lengths equal to a and b.
21. Through a point not on a given line there passes not more than one parallel to the line.
22. Parallel lines are everywhere equidistant from one another.
23. There exists a convex quadrilateral whose angle sum is 360 degrees.
24. Any two parallel lines have a common perpendicular.

In the next two sections of these notes we shall investigate some of these equivalences.

We shall conclude this section with a summary of the results from Euclidean geometry whose proofs do not require the Fifth Postulate.

*How much can one prove **without** the Fifth Postulate?*

Clearly the first step in studying the role of the Fifth Postulate is to understand which results in Euclidean geometry do not depend logically upon that statement. There are numerous examples of proofs in classical Euclidean geometry which do not depend upon the Fifth Postulate or an equivalent statement, but there are also other cases where frequently given proofs depend upon these assumptions but it is also possible to give proofs which do not. Therefore we list here some geometric results which hold regardless of whether or not the Fifth Postulate is true but might seem to require this assumption; proofs these results for neutral geometry can be found in previous lectures or exercises from this course. In each of these results, we assume that everything lies in some neutral planne $\mathbb{P}$ or a neutral **3** – space.

Pasch's Theorem. *Suppose we are given* $\triangle$**ABC** *and a line* **L** *in the same plane as the triangle such that* **L** *meets the open side* **(AB)** *in exactly one point. Then either* **L** *passes through* **C** *or else* **L** *has a point in common with* **(AC)** *or* **(BC).**

Crossbar Theorem. *Let* **A, B, C** *be noncollinear points, and let* **D** *be a point in the interior of* $\angle$**CAB.** *Then the segment* **(BC)** *and the open ray* **(AD** *have a point in common.*

Trichotomy Principle. *Let* **A** *and* **B** *be distinct points, and let* **C** *and* **D** *be two points on the same side of* **AB**. *Then exactly one of the following is true:*

(1) **D** *lies on* **(BC** (*equivalently, the open rays* **(BC** *and* **(BD** *are equal*).

(2) **D** *lies in* **Int** $\angle$**ABC.**

(3) **C** *lies in* **Int** $\angle$**ABD.**

Isosceles Triangle Theorem. *In* $\triangle$**ABC,** *one has* $|AB| = |AC|$ *if and only if* $|\angle ABC| = |\angle ACB|$.

Equilateral Triangle Theorem. *In* $\triangle$**ABC,** *one has* $|AB| = |AC| = |BC|$ *(the triangle is* ***equilateral****) if and only if one has* $|\angle ABC| = |\angle ACB| = |\angle BAC|$ *(the triangle is* ***equiangular****).*

Existence of Perpendculars I. *Let* **L** *be a line, let* **A** *be a point of* **L,** *and let* $\mathbb{P}$ *be a plane containing* **L.** *Then there is a unique line* **M** *in* $\mathbb{P}$ *such that* $A \in M$ *and* $L \perp M$.

Existence of Perpendculars II. *Let* **L** *be a line, and let* **A** *be a point* ***not*** *on* **L.** *Then there is a unique line* **M** *such that* $A \in M$ *and* $L \perp M$.

Parallels and Perpendiculars. *Suppose that* **L, M** *and* **N** *are three lines in the plane* $\mathcal{P}$ *such that we have* $L \perp M$ *and* $M \perp N$. *Then we also have* $L \,||\, N$.

Perpendicular Bisector Theorem. *Let* **A** *and* **B** *be distinct points, let* $\mathbb{P}$ *be a plane containing them, suppose that* **D** *is the midpoint of* **[AB],** *and let* **M** *be the unique perpendicular to* **AB** *at* **D** *in the plane* $\mathbb{P}$. *Then a point* $X \in \mathbb{P}$ *lies on* **M** *if and only if* $|XA| = |XB|$.

Exterior Angle Theorem. *Suppose we are given triangle* $\triangle$**ABC,** *and let* **D** *be a point such that* **B*C*D.** *Then* $|\angle ACD|$ *is greater than both* $|\angle ABC|$ *and* $|\angle BAC|$.

<u>Corollary 1.</u> *If* $\triangle$**ABC** *is an arbitrary triangle, then the sum of any two of the angle measures* $|\angle ABC|$, $|\angle BCA|$ *and* $|\angle CAB|$ *is less than* $180°$. *Furthermore, at least two of these angle measures must be less than* $90°$.

<u>Corollary 2.</u> *Suppose we are given triangle* $\triangle$**ABC,** *and assume that the two angle measures* $|\angle BCA|$ *and* $|\angle CAB|$ *are less than* $90°$. *Let* **D** $\in$ **AC** *be such that* **BD** *is perpendicular to* **AC.** *Then* **D** *lies on the open segment* **(AC).**

<u>Corollary 3.</u> *Suppose we are given triangle* $\triangle$**ABC.** *Then at least one of the following three statements is true*:

(1) *The perpendicular from* **A** *to* **BC** *meets the latter in* **(BC).**
(2) *The perpendicular from* **B** *to* **CA** *meets the latter in* **(CA).**
(3) *The perpendicular from* **C** *to* **AB** *meets the latter in* **(AB).**

<u>Classical Triangle Inequality.</u> *In* $\triangle$**ABC** *we have* $|AC| < |AB| + |BC|$.

<u>Half of the Alternate Interior Angle Theorem.</u> *Suppose we are given two lines* **L** *and* **M** *together with a transversal* **N** *meeting the lines in different points. If the measures of one pair of alternate interior angles are equal, then the lines* **L** *and* **M** *are parallel.*

<u>Half of the Corresponding Angle Theorem.</u> *Suppose we are given the setting and notation above. If the measures of one pair of corresponding angles are equal, then the lines* **L** *and* **M** *are parallel.*

<u>A.A.S. Triangle Congruence Theorem.</u> *Suppose we have two ordered triples of noncollinear points* **(A, B, C)** *and* **(D, E, F)** *satisfying the conditions* $|BC| = |EF|$, $|\angle ABC| = |\angle DEF|$, *and* $|\angle CAB| = |\angle FDE|$. *Then we have* $\triangle ABC \cong \triangle DEF$.

<u>Theorem on Diagonals of a Convex Quadrilateral.</u> *Suppose that* **A, B, C** *and* **D** *form the vertices of a convex quadrilateral. Then the open diagonal segments* **(AC)** *and* **(BD)** *have a point in common.*

<u>H.S. Right Triangle Congruence Theorem</u> *Suppose we have two ordered triples of noncollinear points* **(A, B, C)** *and* **(D, E, F)** *satisfying the conditions* $|\angle ABC| = |\angle DEF| = 90°$, $|AC| = |DF|$, *and* $|BC| = |EF|$. *Then* $\triangle ABC \cong \triangle DEF$.

Empirical questions

There are some immediate questions whether Playfair's Postulate actually "reflects physical reality." The key issues are summarized in a passage on page 123 in the book, ***Mathematics: The Science of Patterns***, by K. Devlin (Owl Books, 1996):

> Suppose you drew a line on a sheet of paper and marked a point not on the line. You are now faced with the task of showing that there is one and only one parallel to the given line that passes through the chosen point. But there are obvious difficulties here. For one, no matter how fine the point of your pencil, the lines you draw still have a definite thickness, and how do you know where the [supposedly] actual ***lines*** are? Second, in order to check that your second line is in fact parallel to the first, you would have to extend both lines indefinitely, which is [physically] not possible. Certainly, you can draw many lines through the given point that do not meet the given line ***on the paper***.

In some sense, the difference between Euclid's Fifth Postulate and Playfair's Postulate is that the former assumes gives a condition that two lines will eventually meet at some possibly remote location, but the latter assumes there are lines that will never meet. If one prefers to avoid questions whether two lines might meet at locations that are physically inaccessible, then the option of assuming an equivalent statement about a bounded portion of space may seem promising. The discussion in Devlin's book also addresses this.

> Thus, Playfair's Postulate is not really suitable for experimental verification. How about the triangle postulate [namely, the angle sum of some triangle is 180 degrees]? Certainly, verifying this postulate does not require extending the lines indefinitely; it can all be done "on the paper." Admittedly, it is likely that no one has any strong intuition concerning the angle sum of a triangle being 180 degrees, the way we do about the existence of unique parallels, but since the two statements are entirely equivalent, the absence of any supporting intuition does not affect the validity of the triangle approach.

If we want to test the statement about angle sums experimentally, we run into immediate problems. First of all, the unavoidability of experimental errors means it is effectively impossible to draw any firm conclusions that the angle sum is ***exactly*** 180 degrees. In contrast, it is conceivable that experimental measurements could show that the angle sum is ***NOT*** equal to 180 degrees, with the deviation exceeding any possible experimental error. There are frequently repeated assertions that Gauss actually tried to carry out such an experiment but his results were inconclusive because the value of 180 degrees was within the expected margin of error. However, there is no hard evidence to confirm such stories.

3 : Neutral geometry

In this section we shall investigate some of the logical equivalences in the list from the previous section. These will play an important role in Section 4.

We have noted that a great deal of work was done in the 17th and 18th century to study classical geometry without using Euclid's Fifth Postulate; early in the 19th century this subject was called ***absolute geometry***, but in modern texts it is generally known as ***neutral geometry***. In this section we shall develop some aspects of this subject more explicitly than in the preceding section. We begin by recalling the formal definition of a neutral geometry.

Definition. A ***neutral plane*** is given by data $(\mathbb{P}, \mathcal{L}, d, \alpha)$ which satisfy all the axioms in Moise except (possibly) Playfair's Postulate or an equivalent statement such as Euclid's Fifth Postulate. Usually we simply denote a neutral plane by its underlying set of points $\mathbb{P}$.

In this setting, the efforts to prove the Fifth Postulate can be restated as follows:

INDEPENDENCE PROBLEM FOR THE FIFTH POSTULATE. *If $\mathbb{P}$ is a neutral plane, is Playfair's Postulate true in $\mathbb{P}$?*

It is important to note that ***all proofs for neutral planes must be done synthetically*** (without coordinates) because Playfair's Postulate is essentially built into the coordinate approach to Euclidean geometry.

The Saccheri – Legendre Theorem

One of the cornerstones of neutral and non – Euclidean geometry is the study of the following issue:

ANGLE SUMS OF TRIANGLES. *Given a triangle $\triangle ABC$, what can we say about the angle sum $|\angle ABC| + |\angle BCA| + |\angle CAB|$ and what geometric information does it carry?*

We know that the angle sum in Euclidean geometry is always $180°$, and as noted in the preceding section this fact is logically equivalent to the Fifth Postulate. On the other hand, we have also seen that the angle sum in spherical geometry is always greater than and that the difference between these quantities is proportional to the area of a spherical triangle. In any case, the angle sum of a triangle was a central object of study in 17th and 18th century efforts to prove the Fifth Postulate.

Most of the arguments below are similar to proofs in high school geometry, with extra attention to questions about order and separation.*However, at several points we need the following properties of real numbers.

*Lecture 14 ended here.