# 3 : Neutral geometry

In this section we shall investigate some of the logical equivalences in the list from the previous section. These will play an important role in Section 4.

We have noted that a great deal of work was done in the 17th and 18th century to study classical geometry without using Euclid's Fifth Postulate; early in the 19th century this subject was called ***absolute geometry***, but in modern texts it is generally known as ***neutral geometry***. In this section we shall develop some aspects of this subject more explicitly than in the preceding section. We begin by recalling the formal definition of a neutral geometry.

**Definition.** A ***neutral plane*** is given by data $(\mathbb{P}, \mathcal{L}, d, \alpha)$ which satisfy all the axioms in Moise except (possibly) Playfair's Postulate or an equivalent statement such as Euclid's Fifth Postulate. Usually we simply denote a neutral plane by its underlying set of points $\mathbb{P}$.

In this setting, the efforts to prove the Fifth Postulate can be restated as follows:

**INDEPENDENCE PROBLEM FOR THE FIFTH POSTULATE.** *If $\mathbb{P}$ is a neutral plane, is Playfair's Postulate true in $\mathbb{P}$?*

It is important to note that ***all proofs for neutral planes must be done synthetically*** (without coordinates) because Playfair's Postulate is essentially built into the coordinate approach to Euclidean geometry.

*The Saccheri – Legendre Theorem*

One of the cornerstones of neutral and non – Euclidean geometry is the study of the following issue:

**ANGLE SUMS OF TRIANGLES.** *Given a triangle $\triangle ABC$, what can we say about the angle sum $|\angle ABC| + |\angle BCA| + |\angle CAB|$ and what geometric information does it carry?*

We know that the angle sum in Euclidean geometry is always $180°$, and as noted in the preceding section this fact is logically equivalent to the Fifth Postulate. On the other hand, we have also seen that the angle sum in spherical geometry is always greater than and that the difference between these quantities is proportional to the area of a spherical triangle. In any case, the angle sum of a triangle was a central object of study in 17th and 18th century efforts to prove the Fifth Postulate.

Most of the arguments below are similar to proofs in high school geometry, with extra attention to questions about order and separation. However, at several points we need the following properties of real numbers.

**Archimedean Law.** *Suppose that* $b$ *and* $a$ *are positive real numbers. Then there is a positive integer* $n$ *such that* $na > b$.■

**Immediate consequence.** *If* $h$ *and* $k$ *are positive real numbers, then there is a positive integer* $n$ *such that* $h/2^n < k$.■

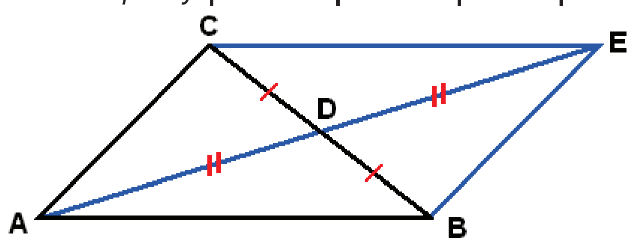
The second statement is proven in the exercises. One informal way of seeing the first statement is to note that the positive number $b/a$ can be written as $x + y$, where $x$ is a nonnegative integer and $y$ lies in the half – open interval $[0, 1)$; one can then take $n = x + 1$.

Our first result on angle sums is a result on finding new triangles with the same angle sums as a given one.

**Proposition 1.** *Suppose that* **A, B, C** *are noncollinear points in the neutral plane* **P**. *Then there exist noncollinear points* **A′, B′, C′** *such that the following hold*:

1. *The angle sums of* △**ABC** *and* △**A′B′C′** *are equal; in other words, we have* |∠**ABC**| + |∠**BCA**| + |∠**CAB**| = |∠**A′B′C′**| + |∠**B′C′A′**| + |∠**C′A′B′**|.
2. *We also have the inequality* |∠**C′A′B′**| ≤ ½|∠**CAB**|.

3.




Same drawing as in the proof of the Exterior Angle Thm.

***Proof.*** Let **D** be the midpoint of **[BC]**, and let **E** be a point on **(AD** satisfying **A∗D∗E** and |**AE**| = 2|**AD**|; it follows that |**AD**| = |**DE**|. We then have △**CDA** ≅ △**BDE** by **S.A.S.**, and therefore it follows that |∠**CAE**| = |∠**AEB**| and |∠**ACB**| = |∠**CBE**|.

**CLAIM:** △**EAB** ***has the same angle sum as*** △**CAB.** — As in the proof of the Exterior Angle Theorem, we know that **E** lies in the interior of ∠**CAB.** Therefore we have |∠**CAB**| = |∠**CAE**| + |∠**EAB**|. Since **C** lies on **(BD** and **D** lies in the interior of ∠**ABE,** we also have |∠**ABE**| = |∠**ABC**| + |∠**CBE**|. Combining this with the triangle congruence from the previous paragraph, we see that

|∠**EAB**| + |∠**ABE**| + |∠**BEA**| =

|∠**EAB**| + |∠**EBC**| + |∠**ABC**| + |∠**BEA**| =

|∠**EAB**| + |∠**ACB**| + |∠**ABC**| + |∠**CAE**| =

|∠**BCA**| + |∠**ACB**| + |∠**ABC**|.

It follows by a similar argument (reversing the roles of **B** and **C**) that

|∠**ACE**| + |∠**CEA**| + |∠**EAC**| =

$$|\angle ABC| + |\angle BCA| + |\angle CAB|.$$

In other words, both $\triangle EAC$ and $\triangle EAB$ have the same angle sum as $\triangle ABC$.

Since $|\angle CAB| = |\angle CAE| + |\angle EAB|$ and the two summands in the right hand expression are positive, at least one of them is less than or equal to $\frac{1}{2}|\angle CAB|$. Depending upon whether $\angle CAE$ or $\angle EAB$ has this property, take $\triangle A'B'C'$ to be $\triangle AEC$ or $\triangle ABE$. ■

**Corollary 2.** *If $\varepsilon > 0$ is a positive real number, then there is a triangle $\triangle A'B'C'$ which has the same angle sum as $\triangle ABC$ but $|\angle C'A'B'| < \varepsilon$.*

***Proof.*** Repeated application of Proposition 1 shows that for each positive integer $n$ there is a triangle $\triangle A_nB_nC_n$ with the same angle sum as $\triangle ABC$ but such that $|\angle C_nA_nB_n| \leq |\angle CAB|/2^n$. Since we know that the right hand side is less than $\varepsilon$ for $n$ sufficiently large, the corollary follows. ■

The preceding results allow us to prove half of the usual Euclidean theorem on angle sums:

**Theorem 3 (Saccheri – Legendre Theorem)** *If* **A, B, C** *are noncollinear points in a neutral plane* **P**, *then* $|\angle ABC| + |\angle BCA| + |\angle CAB| \leq 180°$.

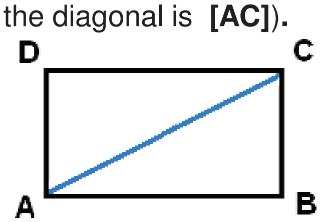
***Proof.*** Suppose we have a triangle $\triangle ABC$ for which the angle sum is strictly greater than $180°$, and write $|\angle ABC| + |\angle BCA| + |\angle CAB| = 180° + \delta°$, where $\delta$ is positive. By Corollary 2 there is a triangle $\triangle A'B'C'$ which has the same angle sum as $\triangle ABC$ but also satisfies $|\angle C'A'B'| < \frac{1}{2}\delta$. It then follows that

$$|\angle B'C'A'| + |\angle A'B'C'| > 180° + \tfrac{1}{2}\delta° > 180°.$$

On the other hand, by a corollary to the Exterior Angle Theorem we also know that the sum of the measures of two vertex angles is always less than $180°$, so we have a contradiction. The problem arises from our assumption that the angle sum of the original triangle is strictly greater than $180°$, and therefore we conclude that the angle sum is at most $180°$. ■

**Corollary 4.** *If* **A, B, C, D** *are the vertices of a convex quadrilateral in a neutral plane* **P**, *then* $|\angle ABC| + |\angle BCD| + |\angle CDA| + |\angle DAB| \leq 360°$.

***Proof.*** The idea is standard; we slice the quadrilateral into two triangles along a diagonal (in the drawing below, the diagonal is **[AC]**).




By the definition of a convex quadrilateral we know that **A** lies in the interior of $\angle BCD$ and **C** lies in the interior of $\angle DAB$, so that $|\angle BCD| = |\angle ACD| + |\angle ACB|$ and likewise $|\angle DAB| = |\angle CAD| + |\angle CAB|$. The Saccheri – Legendre Theorem implies that

$$|\angle ABC| + |\angle BCA| + |\angle CAB| \leq 180°$$
$$|\angle ADC| + |\angle DCA| + |\angle DAB| \leq 180°$$

and if we combine these with the sum identities in the preceding sentence we obtain

$$|\angle ABC| + |\angle BCD| + |\angle CDA| + |\angle DAB| =$$
$$|\angle ABC| + |\angle BCA| + |\angle CAB| + |\angle ADC| + |\angle DCA| + |\angle DAB| \leq$$
$$180° + 180° = 360°$$

which is the statement of the corollary.■

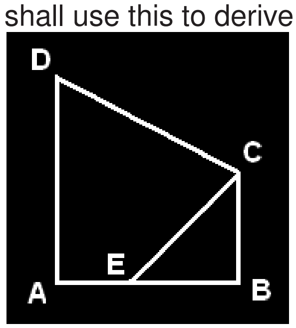
In Section **1** we noted that the angle sum of a triangle is always greater than **180°** degrees in spherical geometry. A unified perspective on neutral and spherical geometry will be discussed in Section **5.**

*Rectangles in neutral geometry*

Rectangles are fundamentally important in both the synthetic and the analytic approaches to Euclidean geometry, so it is not surprising that rectangles and near – rectangles also play an important role in neutral geometry. Before we define the near – rectangles that are studied in neutral geometry, it will be convenient to have a simple criterion for recognizing certain special convex quadrilaterals.

**Proposition 5.** *Let* **A, B, C, D** *be four points in a neutral plane* $\mathbb{P}$ *such that no three are collinear and* **AB** *is perpendicular to* **BC** *and* **AD.** *If* **C** *and* **D** *lie on the same side of* **AB,** *then* **A, B, C, D** *form the vertices of a convex quadrilateral.*

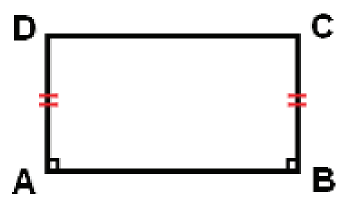
***Proof.*** The lines **BC** and **AD** are parallel since they are perpendicular to the same line (and they are unequal because the four given points are noncollinear). If we combine this with the condition in the second sentence of the proposition, we see that it is only necessary to prove that **A** and **B** lie on the same side of **CD,** so let us suppose this is false. In this case it follows that the segment **(AB)** and the line **CD** have a point **E** in common. We shall use this to derive a contradiction.




Since **C** and **D** lie on the same side of **AB,** it follows that the rays **[EC** and **[ED** are equal (in the picture it appears that the points are not collinear, but this is not a problem since we are trying to derive a contradiction). Therefore we have $|\angle AED| + |\angle CEB| = 180°$. On the other hand, two applications of the Exterior Angle Theorem imply that $|\angle AED| = |\angle AEC| > |\angle CEB| = 90°$ and $|\angle CEB| = |\angle DEB| > |\angle DAB| = 90°$, which in turn implies that $|\angle AED| + |\angle CEB| > 180°$. Thus we have a contradiction; the source of the contradiction was the assumption that **A** and **B** do not lie on the same side of **CD,** and therefore **A** and **B** must lie on the same side of **CD;** as noted before, this is what we needed to complete the proof. ■

The following analogs of rectangles in neutral geometry were studied extensively by Saccheri, but they also appear in earlier mathematical writings of Omar Khayyam.

**Definition.** Let **A, B, C, D** be four points in a neutral plane $\mathbb{P}$ such that no three are collinear. We shall say that these points (in the given order) ***form the vertices of a Saccheri quadrilateral with base*** **AB** provided that **(1)** the line **AB** is perpendicular to **BC** and **AD,** **(2)** the points **C** and **D** lie on the same side of **AB,** **(3)** the lengths of the sides **[BC]** and **[AD]** are equal — in other words, we have $|AD| = |BC|$. In some books and articles, such a figure is called an ***isosceles birectangle.***




By the previous proposition we know that **A, B, C, D** are the vertices of a convex quadrilateral, and we say that this quadrilateral **☐ABCD** is a ***Saccheri quadrilateral*** with base **AB.** — The reason for considering Saccheri quadrilaterals is that it is always possible to construct such figures in a neutral plane, and in fact if $p$ and $q$ are arbitrary positive real numbers then there is a Saccheri quadrilateral **☐ABCD** with base **AB** such that $|AD| = |BC| = p$ and $|AB| = q$ (the proof is left to the exercises).

Of course, a rectangle in Euclidean geometry is a Saccheri quadrilateral, and the next result describes some common properties of Euclidean rectangles that also hold for Saccheri quadrilaterals in neutral geometry.

**Proposition 6.** *If* **A, B, C, D** *are the vertices of a Saccheri quadrilateral with base* **AB,** *then* $|AC| = |BD|$ *and* $|\angle CDA| = |\angle DCB| \leq 90°$. *Furthermore, the line joining the midpoints of* **[AB]** *and* **[CD]** *is perpendicular to both* **AB** *and* **CD.**

A proof of this result is sketched in the exercises. ■

The second part of the previous result implies that the line joining the midpoints of the top and base split the Saccheri quadrilateral into to near – rectangles with a different definition.

**Definition.** Let **A, B, C, D** be four points in a neutral plane $\mathbb{P}$ such that no three are collinear. We shall say that these points (in the given order) ***form the vertices of a Lambert quadrilateral*** provided three of the four lines **AB, BC, CD, DA** are perpendicular to each other (hence there are right angles at three of the four vertices).

In this case it is also straightforward to see that **A, B, C, D** are the vertices of a convex quadrilateral. Suppose, say, that we have right angles at **A, B** and **C.** As in the case of Saccheri quadrilaterals we know that **AD** is parallel to **BC,** but now we also know that **AB** and **CD** are parallel because they are both perpendicular to **BC.** Predictably, under these conditions we say that the quadrilateral ☐**ABCD** is a ***Lambert quadrilateral.*** — It follows that if ☐**XYZW** is a Saccheri quadrilateral with base **XY,** then the line joining the midpoints of **[XY]** and **[ZW]** splits ☐**XYZW** into two Lambert quadrilaterals (this is shown in one of the exercises).

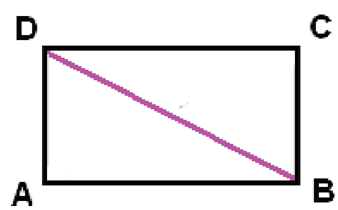
It is also fairly easy to construct Lambert quadrilaterals in a neutral plane. In fact, if $p$ and $q$ are arbitrary positive real numbers then there is a Lambert quadrilateral ☐**ABCD** with right angles at **A, B,** and **C** such that $|AD| = p$ and $|AB| = q$ (the proof is again left to the exercises). By Corollary 4 we know that $|\angle CDA| \leq 90°$.

Having defined types of near – rectangles that exist in every neutral plane, we can now give a neutral – geometric definition of "genuine" rectangles:

<u>**Definition.**</u> Let **A, B, C, D** be four points in a neutral plane $\mathbb{P}$ such that no three are collinear. We shall say that these points ***form the vertices of a <u>rectangle</u>*** (<u>*in the given order*</u>) provided the four lines **AB, BC, CD, DA** are perpendicular to each other at **B, C, D** and **A.** As before, the points **A, B, C, D** (in the given order) are the vertices of a convex quadrilateral, and we say that the quadrilateral ☐**ABCD** is a ***rectangle.***

**Since we do not have Playfair's Postulate at our disposal, we must be careful about not using results from Euclidean geometry which depend upon this postulate when we prove theorems about rectangles in neutral geometry,** and frequenty we need new synthetic proofs for extremely familiar facts. Here are a few examples.

<u>**Theorem 7.**</u> *If* **A, B, C, D** *are the vertices of a rectangle, then* $|AB| = |CD|$ *and* $|AD| = |BC|$. *Furthermore, we have* $\triangle DAB \cong \triangle BCD$, *and the angle sums for these triangles are equal to* $180°$.




<u>***Proof.***</u> In order to simplify some of the algebraic manipulations, it is helpful to denote various angle measures by letters:

$$|\angle ADB| = \alpha, \quad |\angle DBA| = \beta, \quad |\angle DBC| = \gamma, \quad |\angle BDC| = \delta$$

Since **D** and **B** lie in the interiors of $\angle CBA$ and $\angle CDA$ respectively, it follows that

$$\alpha + \delta = 90° = \beta + \gamma.$$

On the other hand, the Saccheri – Legendre Theorem implies and the perpendicularity conditions imply that $\alpha + \beta \leq 90°$ and $\gamma + \delta \leq 90°$. These imply that the sum of $\alpha, \beta, \gamma, \delta$ is less than or equal to $180°$, while the displayed equations imply that the sum of these four numbers is equal to $180°$. If either of the inequalities were strict, then the sum would be strictly less than $180°$, and hence we have a pair of equations $\alpha + \beta = 90° = \gamma + \delta$. Thus we have a system of two linear equations for $\alpha, \beta, \gamma, \delta$, and the solutions of this system are given by $\alpha = \beta$ and $\delta = \gamma$. The conclusion about angle sums for the two triangles $\triangle$**DAB** and $\triangle$**BCD** follows immediately from this.

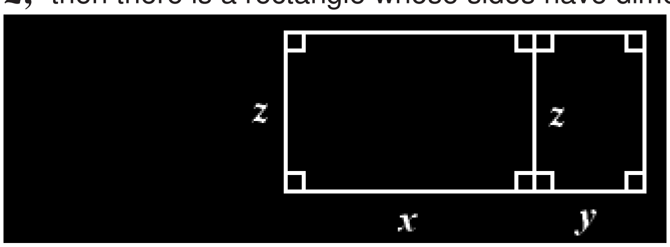
Furthermore, it also follows that $\triangle$**DAB** $\cong$ $\triangle$**BCD** by **S.A.S.** The remaining conclusions $|AB| = |CD|$ and $|AD| = |BC|$ follow from this triangle congruence. ■

**IMPORTANT NOTE.** Although we have defined the concept of a rectangle for an arbitrary neutral plane, ***we do not necessarily know if there are any rectangles at all in a given neutral plane* P *unless we know that Playfair's Postulate holds in* P.** The logical relationship between the Euclidean Parallel Postulate and the existence of rectangles was a central point in the writings of A. – C. Clairaut (1713 – 1765) on classical Euclidean geometry. It turns out that the existence of even one rectangle in $\mathbb{P}$ has extremely strong consequences, most of which arise from the following result.

**Theorem 8.** *Suppose there is at least one rectangle in a given neutral plane $\mathbb{P}$. Then for every pair of positive real numbers $p$ and $q$ there is a rectangle* □**ABCD** *such that* $|AB| = |CD| = p$ *and* $|AD| = |BC| = q$.

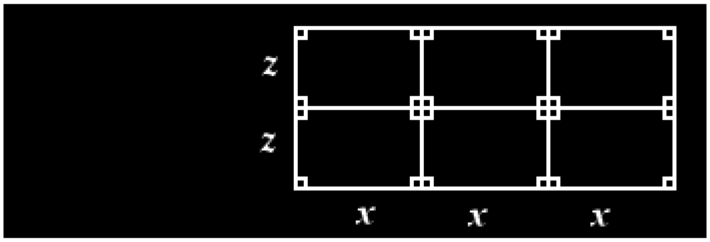
The proof of this theorem is fairly long and has several steps.

1. A splicing construction, which shows if there is a rectangle whose sides have dimensions $x$ and $z$ and a rectangle whose sides have dimensions $y$ and $z$, then there is a rectangle whose sides have dimensions $(x + y)$ and $z$.

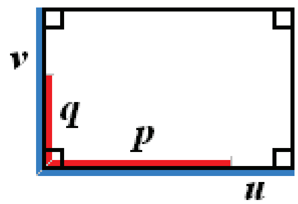



2. Repeated application of the splicing construction to show that if there is a rectangle whose sides have dimensions $x$ and $z$ and we are given positive integers $m$ and $n$, then there is a rectangle whose sides have dimensions $mx$

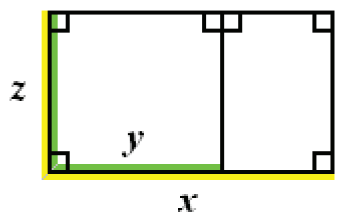
and $nz$. In the drawing below, $m = 3$ and $n = 2$.




3. Combining the previous two steps with the Archimedean Property of real numbers to show that if a rectangle exists, then there is a rectangle whose sides have dimensions $u$ and $v$, where $u > p$ and $v > q$.




4. A trimming – down construction, which shows that if there is a rectangle whose sides have dimensions $x$ and $z$ and $y$ is a positive number less than $x$, then there is a rectangle whose sides have dimensions $y$ and $z$. Two applications of this combine with the third step to prove Theorem 8.




The proofs for several of these steps are quite lengthy in their own right. Therefore we shall now move forward, with the details in an Appendix to this section. ■

*The All – or – Nothing Theorem for angle sums*

The preceding result on rectangles has an immediate consequence for angle sums of triangles.

**Theorem 9.** *If a rectangle exists in a neutral plane $\mathbb{P}$, then every right triangle in $\mathbb{P}$ has an angle sum equal to* $180°$.

***Proof.*** Suppose we are given right triangle $\triangle ABC$ with a right angle at $B$. By the preceding result there is a rectangle $\square WXYZ$ such that $|AB| = |WX|$ and $|BC| = |XY|$. By S.A.S. we have $\triangle ABC \cong \triangle WXY$; in particular, the angle sums of these triangles are equal. On the other hand, the proof of Theorem 7 implies

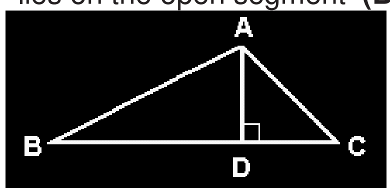
**Thms. 9 and 10 stated without proof at the end of Lecture 15. Details for the proof of Thm. 8 in the file lecture15a.pdf**

that the angle sum of $\triangle$**WXY** is equal to $180°$, so the same must be true for $\triangle$**ABC.** ■

This result extends directly to arbitrary triangles in the neutral plane $\mathbb{P}$.

**Theorem 10.** *If a rectangle exists in a neutral plane $\mathbb{P}$, then every triangle in $\mathbb{P}$ has an angle sum equal to $180°$.*

***Proof.*** The idea is simple; we split the given triangle into two right triangles and apply the preceding result. By a corollary to the Exterior Angle Theorem, we know that the perpendicular from one vertex of a triangle meets the opposite side in a point between the other two vertices (in particular, we can take the vertex opposite the longest side). Suppose now that the triangle is labeled $\triangle$**ABC** so that the foot **D** of the perpendicular from **A** to **BC** lies on the open segment **(BC).**




We know that **D** lies in the interior of $\angle$**BAC,** and therefore we have

$$|\angle BAD| + |\angle DAC| = |\angle BAC|.$$

By the previous result on angle sums for right triangles, we also have

$$|\angle BAD| + |\angle ADB| = 90° = |\angle DAC| + |\angle ACD|$$

and if we combine all these equations we find that

$$|\angle ABC| + |\angle BCA| + |\angle CAB| = 180°$$

which is the desired conclusion. ■

There is also a converse to the preceding two results.

**Theorem 11.** *If a neutral plane $\mathbb{P}$ contains at least one triangle whose angle sum is equal to $180°$, then $\mathbb{P}$ contains a rectangle.*

***Proof.*** The idea is to reverse the preceding discussion; we first show that under the given conditions there must be a right triangle whose angle sum is equal to $180°$, and then we use this to show that there is a rectangle.

**FIRST STEP:** If there is a triangle whose angle sum is $180°$, then there is also a right triangle with this property.

Given a triangle whose angle sum is $180°$, as in the previous result we label the vertices **A, B, C** so that the foot of the perpendicular from **A** to **BC** lies on the open segment **(BC).** Reasoning once again as in the proof of Theorem 10 we find

$$\text{Angle sum}(\triangle ABD) + \text{Angle sum}(\triangle ADC) =$$

$$\text{Angle sum}(\triangle ABC) + 180° = 180° + 180° = 360°.$$