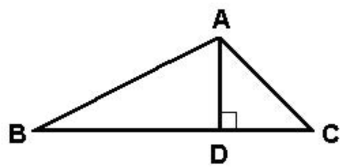
that the angle sum of $\triangle$**WXY** is equal to $180°$, so the same must be true for $\triangle$**ABC.** ■

This result extends directly to arbitrary triangles in the neutral plane $\mathbb{P}$.

<u>Theorem 10.</u> *If a rectangle exists in a neutral plane $\mathbb{P}$, then every triangle in $\mathbb{P}$ has an angle sum equal to $180°$.*

<u>Proof.</u> The idea is simple; we split the given triangle into two right triangles and apply the preceding result. By a corollary to the Exterior Angle Theorem, we know that the perpendicular from one vertex of a triangle meets the opposite side in a point between the other two vertices (in particular, we can take the vertex opposite the longest side). Suppose now that the triangle is labeled $\triangle$**ABC** so that the foot **D** of the perpendicular from **A** to **BC** lies on the open segment **(BC).**



We know that **D** lies in the interior of $\angle$**BAC,** and therefore we have

$$|\angle BAD| + |\angle DAC| = |\angle BAC|.$$

By the previous result on angle sums for right triangles, we also have

$$|\angle BAD| + |\angle ADB| = 90° = |\angle DAC| + |\angle ACD|$$

and if we combine all these equations we find that

$$|\angle ABC| + |\angle BCA| + |\angle CAB| = 180°$$

which is the desired conclusion. ■

There is also a converse to the preceding two results.

<u>Theorem 11.</u> *If a neutral plane $\mathbb{P}$ contains at least one triangle whose angle sum is equal to $180°$, then $\mathbb{P}$ contains a rectangle.*

<u>Proof.</u> The idea is to reverse the preceding discussion; we first show that under the given conditions there must be a right triangle whose angle sum is equal to $180°$, and then we use this to show that there is a rectangle.

<u>FIRST STEP:</u> If there is a triangle whose angle sum is $180°$, then there is also a right triangle with this property.

Given a triangle whose angle sum is $180°$, as in the previous result we label the vertices **A, B, C** so that the foot of the perpendicular from **A** to **BC** lies on the open segment **(BC).** Reasoning once again as in the proof of Theorem 10 we find

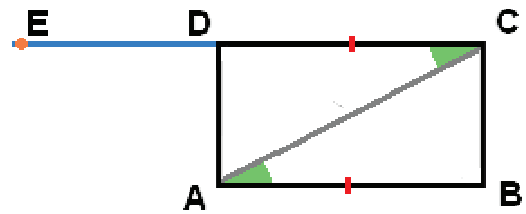
$$\text{Angle sum}(\triangle ABD) + \text{Angle sum}(\triangle ADC) =$$

$$\text{Angle sum}(\triangle ABC) + 180° = 180° + 180° = 360°.$$

Since each of the summands on the left hand side is at most $180°$, it follows that each must be equal to $180°$, (if either were strictly less, then the left side would be less than $360°$). Thus the two right triangles $\triangle$**ABD** and $\triangle$**ADC** have angle sums equal to $180°$.

SECOND STEP: If there is a right triangle whose angle sum is $180°$, then there is also a rectangle.

Once again the idea is simple. We shall construct another right triangle with the same hypotenuse to obtain a rectangle. Suppose that $\triangle$**ABC** is the right triangle whose angle sum is equal to $180°$, and that the right angle of this triangle is at **B.**



By the Protractor Postulate there is a unique ray **[CE** such that **(CE** is on the side of **AC** opposite **B** and $|\angle ECA| = |\angle BAC|$. Take **D** to be the unique point on **(CE** such that $|AB| = |CD|$. Then Theorem **7** and **S.A.S.** imply that $\triangle BAC \cong \triangle DCA$. In particular, we have $|\angle DAC| = |\angle BCA|$ and $|\angle ADC| = |\angle ABC|$.

It follows that **AD** and **DC** are perpendicular, so we know there are right angles at **B** and **D.** Furthermore, the Alternate Interior Angle Theorem (more correctly, *the half which is valid in neutral geometry*) implies that the lines **AB** and **CD** are parallel, and likewise the same result and the triangle congruence imply that **AD** and **BC** are parallel. As in the discussion of Lambert quadrilaterals, these conditions imply that **A, B, C, D** form the vertices of a convex quadrilateral. We shall use this to prove that there are also right angles at **A** and **C.**

Since we now know we have a convex quadrilateral, it follows that **A** and **C** lie in the interiors of $\angle$**BCD** and $\angle$**DAB** respectively. Therefore we have

$$|\angle BCD| = |\angle ACD| + |\angle ACB| = |\angle BAC| + |\angle ACB| = 90°$$

where the last equation holds because of our assumption about the angle sum of the right triangle $\triangle$**ABC.** Thus we know that there also is a right angle at the vertex **C.** But we also have

$$|\angle BAD| = |\angle BAC| + |\angle BCA| = |\angle ACD| + |\angle BCA| = 90°$$

where the final equation this time follows because we have shown there is a right angle at **C.** Thus we see that there is also a right angle at **A** and therefore we have a rectangle.■

This brings us to the main result of this section.

<u>Theorem 12. (All – or – Nothing Theorem)</u> *In a given neutral plane* $\mathbb{P}$, **EITHER** ***<u>every triangle</u>*** *has an angle sum is equal to* $180°$ **OR ELSE** ***<u>no triangle</u>*** *has an angle sum equal to* $180°$. *In the second case the angle sum of every triangle is strictly less than* $180°$.

<u>Proof.</u> This is mainly a matter of sorting through the preceding results. If one triangle has an angle sum equal to $180°$, then by Theorem 11 a rectangle exists, and in that case Theorem 10 implies that every triangle has angle sum equal to $180°$. Therefore it is impossible to have a neutral plane in which some triangles have angle sums equal to $180°$ but others do not. Finally, by the Saccheri – Legendre Theorem we know that if no triangle has angle sum equal to $180°$ then every triangle must have an angle sum that is strictly less than $180°$. ■

The path to hyperbolic geometry

> The sum of the three angles of a plane triangle cannot be greater than $180°$ … But the situation is quite different in the second part — that the sum of the angles cannot be less than $180°$; this is the critical point, the reef on which all the wrecks occur.
>
> C. F. Gauss, ***Letter to F. (W.) Bolyai***

> When you have eliminated the impossible, whatever remains, however improbable [it may seem], must be the truth.
>
> A. C. Doyle (1859 – 1930), ***Sherlock Holmes — Sign of the Four***

In some respects, the results of this section provide reasons to be optimistic about finding a proof of Euclid's Fifth Postulate in an arbitrary neutral plane. First of all, the results on rectangles and angle sums show that Playfair's Postulate is equivalent to statements that look much weaker (for example, the existence of ***just one rectangle*** or ***just one triangle whose angle sum is*** $180°$). Furthermore, the results suggest that the negation of Playfair's Postulate leads to consequences which seem extremely strange and perhaps even unimaginable. However, as Gauss indicated in his letter, no one was able to overcome the final hurdle and give a complete proof of Euclid's Fifth Postulate from the other axioms for Euclidean geometry. Although the efforts to prove Euclid's Fifth Postulate did not lead to the proof, the best work on the problem provided very extensive, and in some cases nearly complete, information on strange things that would happen if one assumes that the Fifth Postulate is false. We shall examine some of these phenomena in the remaining sections of this unit.

Ultimately these considerations led to a viewpoint expressed in another quotation from Gauss' correspondence:

> The theorems of this geometry appear to be paradoxical and, to the uninitiated, absurd; but calm, steady reflection reveals that they contain nothing at all impossible. (***Letter to Taurinus***, 1824; one should compare this to the Sherlock Holmes quotation given above.)

Before Gauss, some mathematicians (for example, Klügel) had speculated that a proof of the Fifth Postulate might be out of reach. However, Gauss (and to a lesser extent a contemporaries like Schweikart and Taurinus) took things an important step further, concluding that the negation of the Fifth Postulate yields a geometrical system which is very different from Euclidean geometry in some respects but has ***exactly the same degree of logical validity*** (compare also the passage from the letter to Olbers at the beginning of this unit). Working independently of Gauss, J. Bolyai (1802 – 1860) and N. I. Lobachevsky (1792 – 1856) reached the same conclusions as Gauss (each one independently of the other), which Bolyai summarized in a frequently repeated quotation:

Out of nothing I have created a strange new universe.

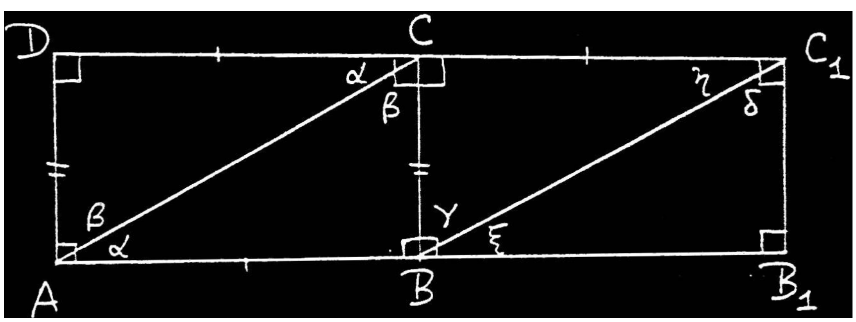
Both Bolyai and Lobachevsky took everything one important step further than Gauss by publishing their conclusions, and for this reason they share credit for the first published recognition of hyperbolic geometry as a mathematically legitimate subject.

The appendix was already covered in L15. Skip to the end of the section.

Appendix to Section 3: Proof of Theorem 8

The major steps in the argument will be presented as lemmas.

Lemma 8A. (Splicing Property). *Suppose that* $\square$**ABCD** *is a rectangle, and let* $\mathbf{C_1} \in$ **(DC** *be a polnt such rhat* $|\mathbf{DC_1}| = 2|\mathbf{DC}|$. *Let* $\mathbf{B_1}$ *be the foot of the perpendicular from* $\mathbf{C_1}$ *to* **AB.** Then $|\angle \mathbf{DC_1B}| = 90°$ *and the points* **A,** $\mathbf{B_1}$, $\mathbf{C_1}$ *and* **D** *(**in that order**) are the vertices of a rectangle.*



Proof. First of all, the lines **AD, BC,** and $\mathbf{B_1C}$, are all parallel to each other because every two of them have a common perpendicular (namely, **AB**). Therefore **AD** and $\mathbf{B_1C_1}$ are contained in the **D** – and $\mathbf{C_1}$ – sides of **BC** respectively. But $|\mathbf{DC_1}| = 2|\mathbf{DC}|$ and $\mathbf{C_1} \in$ **(DC** imply $\mathbf{D*C*C_1}$ is true. This in turn implies that **D** and $\mathbf{C_1}$ are on opposite sides of **BC.** Since **B** is the common point of the lines $\mathbf{AB_1}$ and **BC,** it follows that $\mathbf{A*B*B_1}$ is true.

Since **AD** and $\mathbf{B_1C_1}$ are parallel (they have a common perpendicular), the points $\mathbf{B_1}$ and $\mathbf{C_1}$ lie on the same side of **AD.** Hence **A,** $\mathbf{B_1}$, $\mathbf{C_1}$, and **D** (in that order) form the vertices of a convex quadrilateral. Likewise **B,** $\mathbf{B_1}$, $\mathbf{C_1}$, and **C** form the vertices of a convex quadrilateral. By construction, **S.A.S.** applies to show $\triangle\mathbf{ADC} \cong \triangle\mathbf{BCC_1}$. It follows

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that $|AC| = |BC_1|$, $\gamma = |\angle CBC_1| = |\angle DAC| = \alpha$, and $\eta = |\angle BC_1C| = |\angle ACD|$.

On the other hand, if $\xi = |\angle C_1C\,B_1|$ then

$$\alpha + \beta = 90° = \gamma + \xi .$$

.

Therefore $\alpha = \gamma$ implies $\beta = \xi$.

By **A.A.S.** it follows that $\triangle ABC \cong \triangle BB_1C_1$, and hence $\alpha = \delta = |\angle BC_1\,B_1|$. This implies that $\eta + \delta = 90°$. But then it follows that $|\angle DC_1\,B_1| = \eta + \delta = 90°$. ■

<u>**Lemma 8B.**</u> *If there is a rectangle* $\square$**ABCD** *in the neutral piane under consideration, then for every positive integer* n *there is a rectangle* $\square A'B'C'D'$ *with* $|A'B'| = |C'D'| = n|AB| = n|CD|$ *and* $|A'D'| = |B'C'| = |AD| = |BC|$.

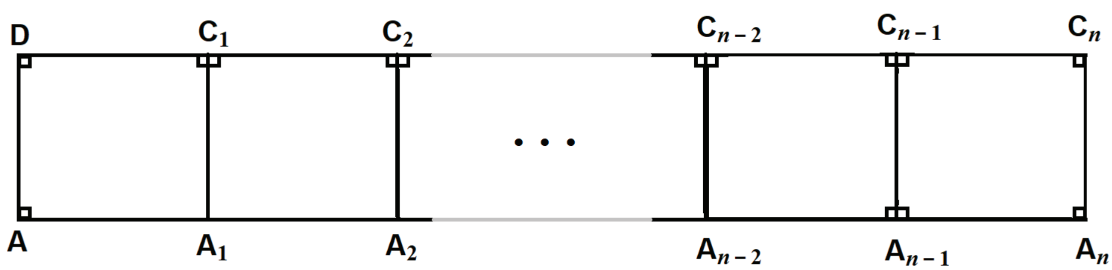
<u>**Proof.**</u> The case $n = 2$ was done in the preceding lemma. Assume by induction that we have $B = A_1, A_2, \ldots, A_{n-1}$ and $C = C_1, C_2, \ldots, C_{n-1}$ such that

$$A = A_0{*}A_1{*}A_2{*} \ldots {*}A_{n-1} \quad\text{and}\quad D = C_0{*}C_1{*}C_2{*} \ldots {*}C_{n-1}$$

(the notation means that $X_p{*}X_q{*}X_r$ if $p < q < r$) and the following additional conditions:

For $k = 1, \ldots, n-1$ we have $|AB| = |CD| = |A_{k-1}A_k| = |C_{k-1}C_k|$.

For $k = 1, \ldots, n-1$ the line C_kA_k is perpendicular to both **AB** and **CD**.



Now apply Lemma 8A to the rectangle $A_{n-2}A_{n-1}C_{n-1}C_{n-2}$ to obtain A_n and C_n such that $A_{n-2}{*}A_{n-1}{*}A_n$ and $C_{n-2}{*}C_{n-1}{*}C_n$, $|AB| = |CD| = |A_{n-1}A_n| = |C_{n-1}C_n|$, and C_nA_n is perpendicular to both **AB** and **CD**. ■

<u>**Corollary 8C.**</u> *If there is a rectangle* $\square$**ABCD** *in the neutral piane under consideration, then for each pair of integers* $n, m > 0$ *there is a rectangle* $\square A'B'C'D'$ *with* $|A'B'| = |C'D'| = n|AB| = n|CD|$ *and* $|A'D'| = |B'C'| = m|AD| = m|BC|$.

<u>**Proof.**</u> First apply Lemma B to get a rectangle $\square A^*B^*C^*D^*$ with $|A^*B^*| = n|AB|$ and $|B^*C^*| = |BC|$. Now apply Lemma B again to get a new rectangle $\square A'B'C'D'$ with

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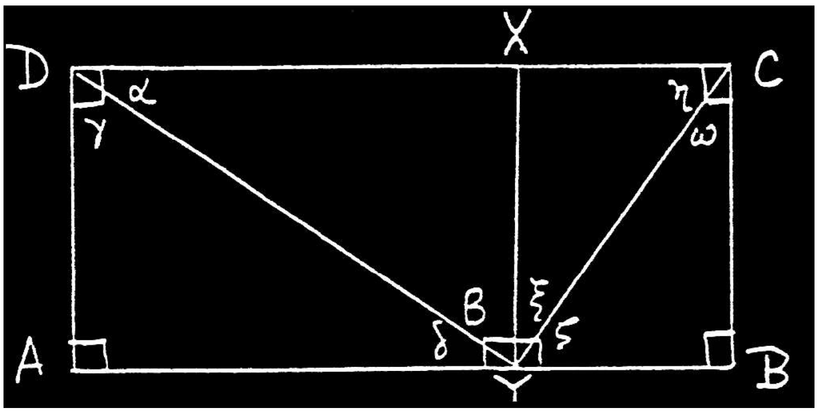
$|A'B'| = |A^*B^*|$ and $|B'C'| = m|B^*C^*|$. It follows that $|A'B'| = |C'D'| = n|AB| = n|CD|$ and $|A'D'| = |B'C'| = m|AD| = m|BC|$. ■

The next lemma allows us to take a large rectangle and trim it to another of srnalfer size.

<u>Lemma 8D.</u> *Let* ☐**ABCD** *be a rectangle*, *let* **X** ∈ **(CD)**, *and let* **Y** *be the foot of the perpendicular from* **X** *to* **AB**. *Then* **A, Y, X,** *and* **D** *(**in that order**) are the vertices of a rectangle.*



<u>Proof.</u> The lines **AD, XY,** and **BC** are all parallel because they are all perpendicular to **AB.** Hence **AD** is contained in the **D** – side of **XY** and **BC** is contained in the **C** – side of **XY.** But **C**∗**X**∗**D** since **X** lies on **(BC)**, and therefore **C** and **D** lie on opposite sides of **XY.** Hence **AD** and **BC** also lie entirely on opposite sides of **XY.** Since **AB** and **XY** meet at **Y,** it follows that **A**∗**Y**∗**B** must be true.

Label the angle measures as indicated in the drawing above:

$\alpha = \lvert\angle YDX\rvert$		$\xi = \lvert\angle CYX\rvert$
$\beta = \lvert\angle DYX\rvert$		$\eta = \lvert\angle XCY\rvert$
$\gamma = \lvert\angle ADY\rvert$		$\zeta = \lvert\angle CYB\rvert$
$\delta = \lvert\angle AYD\rvert$		$\omega = \lvert\angle YCB\rvert$

Since **AD** is parallel to **XY, CD** is parallel to **XY,** and **AB** is parallel to **CD,** it follows that **A, Y, X,** and **D** and **Y, B, X,** and **X** (in these orders) form tbe vertices of a convex quadrilateral. Therefore **D** lies in the interior of ∠**AYX,** **Y** lies in the interior of ∠**ADYX,** **C** lies in the interior of ∠**XYB,** and **Y** lies in the interior of ∠**XCB.** These imply the following four equations:

$\alpha + \gamma = 90°$		$\xi + \zeta = 90°$
$\beta + \delta = 90°$		$\eta + \omega = 90°$

The Saccheri – Legendre Theorem implies the following additional inequalities:

$\xi + \gamma \leq 90°$		$\omega + \zeta \leq 90°$

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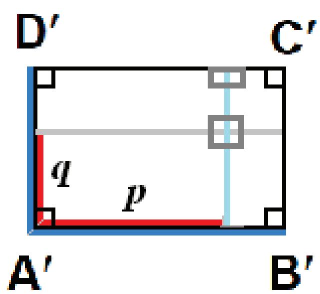
Taken together, these imply $90° \leq \alpha + \beta$ and $90° \leq \xi + \eta$. Therefore the Saccheri – Legendre Theorem irnplies that both $|\angle \mathbf{DXY}|$ and $|\angle \mathbf{CXY}|$ are less than or equal to $90°$. On the other hand, **C∗X∗D** implies that $|\angle \mathbf{DXY}| + |\angle \mathbf{CXY}| = 180°$. This can happen only if both $|\angle \mathbf{DXY}|$ and $|\angle \mathbf{CXY}|$ are equal to $90°$. But this now implies **XY** is perpendicular to **CD,** so that **A, Y, X,** and **D** (in that order) *are* the vertices of a rectangle. ■

__**Proof of Theorem 8.**__ Given rectangle ☐**ABCD** and real numbers $p, q > 0$, find positive integers n and m so that $p < n\,|\mathbf{AB}|$ and $q < m\,|\mathbf{AD}|$. Form a rectangle ☐**A′B′C′D′** with $|\mathbf{A'B'}| = n\,|\mathbf{AB}|$ and $|\mathbf{A'D'}| = q\,|\mathbf{AD}|$.



Let $\mathbf{X} \in (\mathbf{A'B'})$ satisfy $|\mathbf{A'X}| = p < |\mathbf{A'B'}|$, and let **Y** be the foot of the perpendicular from **X** to **C′D′.** Then by Lemma 8D one obtains a rectangle ☐**A′YXD′** with $|\mathbf{A'X}| = p$ and $|\mathbf{A'D'}| = q\,|\mathbf{AD}|$.

Now let $\mathbf{W} \in (\mathbf{A'D'})$ satisfy $|\mathbf{A'W}| = q < |\mathbf{A'D'}|$, and let **V** be the foot of the perpendicular from **Z** to **XY.** Then **A′, Y, V,** and **W** (in that order) are the vertices of a rectangle with $|\mathbf{A'X}| = p$ and $|\mathbf{A'W}| = q$. ■

ADDITIONAL READING FOR THIS SECTION

__**Sections 10.2 and 10.3 of Moise also contain important material that should be read and well understood. Something from them is likely to appear on a quiz or exam.**__

4 : Angle defects and related phenomena

In the previous section we showed that the angle sums of triangles in a neutral plane can behave in one of two very distinct ways. In fact, it turns out that there are essentially only two possible neutral planes, one of which is given by Euclidean geometry and the other of which does not satisfy any of the 24 properties listed in Section 2. The purpose of this section is to study some of these properties for a non – Euclidean plane.

Definition. A neutral plane $(\mathbb{P}, \mathcal{L}, d, \alpha)$ is said to be ***hyperbolic*** if Playfair's Parallel Postulate does **not** hold. In other words,

> there is ***some pair*** **(L, X)**, where **L** is a line in $\mathbb{P}$ and **X** is a point not on **L**, for which there are ***at least two lines through*** **X** which are ***parallel to*** **L**.

The study of hyperbolic planes is usually called **HYPERBOLIC GEOMETRY**.

The name "hyperbolic geometry" was given to the subject by F. Klein (1849 – 1925), and it refers to some relationships between the subject and other branches of geometry which cannot be easily summarized here. Detailed descriptions may be found in the references listed below:

C. F. Adler, ***Modern Geometry: An Integrated First Course*** (2nd Ed.). McGraw – Hill, New York, 1967. ISBN: 0–070–00421–8. [see Section **8.5.3**, pp. 219 – 226]

A. F. Horadam, ***Undergraduate Projective Geometry.*** Pergamon Press, New York, 1970. ISBN: 0–080–17479–5. [see pp. 271 – 272]

H. Levy, ***Projective and Related Geometries.*** Macmillan, New York, 1964. ISBN: 0–000–03704–4. [see Chapter **V**, Section **7**]

A complete and rigorous development of hyperbolic geometry is long and ultimately highly nonelementary, and *it requires a significant amount of input from trigonometry, transcendental functions and differential and integral calculus*. We shall discuss one aspect of the subject with close ties to calculus at the end of this section, but we shall only give proofs that involve "elementary" concepts and techniques.

In the previous section we showed that the angle sum of a triangle in a neutral plane is either always equal to $180°$ or always strictly less than $180°$. We shall begin by showing that the second alternative holds in a hyperbolic plane.

Theorem 1. *In a hyperbolic plane* $\mathbb{P}$ *there is a triangle* $\triangle$**ABC** *such that*

$$|\angle CAB| + |\angle ABC| + |\angle ACB| < 180°.$$

By the results of the preceding section, we immediately have several immediate consequences.

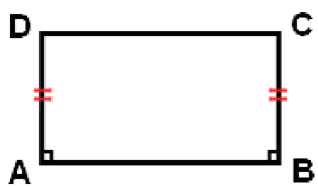
Theorem 2. *In a hyperbolic plane* $\mathbb{P}$*, given an arbitrary triangle* $\triangle$**ABC** *we have*

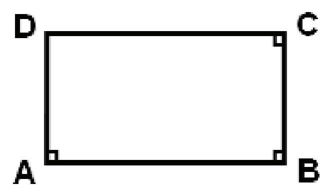
$$|\angle CAB| + |\angle ABC| + |\angle ACB| < 180°.\blacksquare$$

This follows from the All – or – Nothing Theorem in Section 3, and it has further implications for the near – rectangles we have discussed.

<u>**Corollary 3.**</u> *In a hyperbolic plane* $\mathbb{P}$, *suppose that we have a convex quadrilateral* ☐**ABCD** *such that* **AB** *is perpendicular to both* **AD** *and* **BC.**

1. *If* ☐**ABCD** *is a* ***Saccheri quadrilateral*** *with base* **AB** *such that* $|AD| = |BC|$, *then* $|\angle ADC| = |\angle BCD| < 90°$.
2. *If* ☐**ABCD** *is a* ***Lambert quadrilateral*** *such that* $|\angle ABC| = |\angle BCD| = |\angle DAB| = 90°$, *then* $|\angle ADC| < 90°$.





In particular, it follows that <u>**there are NO RECTANGLES in a hyperbolic plane**</u> $\mathbb{P}$.

<u>***Proof of Corollary 3.***</u> If we split each choice of convex quadrilateral into two triangles along the diagonal **[AC]**, then by Theorem 2 we have the following:

$$|\angle CAB| + |\angle ABC| + |\angle ACB| < 180°$$

$$|\angle CAD| + |\angle ADC| + |\angle ACD| < 180°$$

Since is a convex quadrilateral we know that **C** lies in the interior or $\angle$**DAB** and **A** lies in the interior of $\angle$**BCD.** Therefore we have $|\angle DAB| = |\angle DAC| + |\angle CAB|$ and $|\angle BCD| = |\angle ACD| + |\angle ACB|$; if we combine these with the previous inequalities we obtain ***the following basic inequality, which is valid for an arbitrary convex quadrilateral in a hyperbolic plane***:

$$|\angle ABC| + |\angle BCD| + |\angle CDA| + |\angle DAB| =$$

$$|\angle CAB| + |\angle ABC| + |\angle ACB| + |\angle CAD| + |\angle ADC| + |\angle ACD| < 360°$$

To prove the first statement, suppose that ☐**ABCD** is a ***Saccheri quadrilateral***, so that $|\angle ADC| = |\angle BCD|$ by the results of the previous section. Since $|\angle DAB| = |\angle ABC| = 90°$ by Proposition 3.6, the preceding inequality reduces to

$$180° + |\angle BCD| + |\angle CDA| = 180° + 2|\angle BCD| =$$

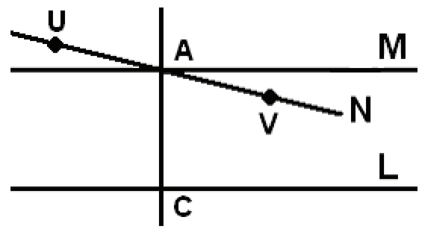
$$180° + 2|\angle CDA| < 360°$$

which implies $|\angle ADC| = |\angle BCD| < 90°$.

To prove the second statement, suppose that ☐**ABCD** is a ***Lambert quadrilateral***, so that $|\angle BCD| = 90°$. Since $|\angle ABC| = |\angle DAB| = 90°$, the general inequality specializes in this case to $270° + |\angle CDA| < 360°$, which implies the desired inequality $|\angle ADC| < 90°$.■

***Proof of Theorem* 1.** In a hyperbolic plane, we know that there is some line **L** and some point **A** not on **L** such that there are at least two parallel lines to **L** which contain **A.**

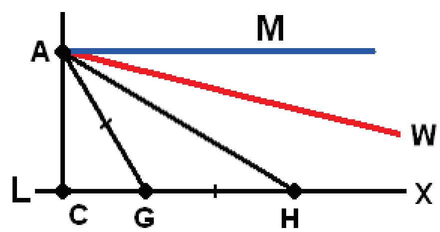
Let **C** be the foot of the unique perpendicular from **A** to **L,** and let **M** be the unique line through **A** which is perpendicular to **AC** in the plane of **L** and **A.** Then we know that **L** and **M** have no points in common (otherwise there would be two perpendiculars to **AC** through some external point). By the choice of **A** and **L** we know that there is a second line **N** through **A** which is disjoint from **L**.



The line **N** contains points **U** and **V** on each side of **AC,** and they must satisfy **U∗A∗V.** Since **N** is not perpendicular to **AC** and $|\angle \mathbf{CAU}| + |\angle \mathbf{CAV}| = 180°$, it follows that one of $|\angle \mathbf{CAU}|$, $|\angle \mathbf{CAV}|$ must be less than $90°$. Choose **W** to be either **U** or **V** so that we have $\theta = |\angle \mathbf{CAW}| < 90°$ (in the drawing above we have **W** = **V**).

The line **L** also contains points on both sides of **AC,** so let **X** be a point of **L** which is on the same side of **AC** as **W.**

<u>**CLAIM:**</u> *If* **G** *is a point of* **(CX,** *then there is a point* **H** *on* **(CX** *such that* **C∗G∗H** *and* $|\angle \mathbf{CHA}| \leq \tfrac{1}{2}|\angle \mathbf{CGA}|$.



To prove the claim, let **H** be the point on **(CX** such that $|\mathbf{CH}| = |\mathbf{CG}| + |\mathbf{GA}|$; it follows that **C∗G∗H** holds and also that $|\mathbf{GH}| = |\mathbf{AG}|$. The Isosceles Triangle Theorem then implies that $|\angle \mathbf{GHA}| = |\angle \mathbf{GAH}|$, and by a corollary to the Saccheri – Legendre Theorem we also have $|\angle \mathbf{CGA}| \geq |\angle \mathbf{GHA}| + |\angle \mathbf{GAH}| = 2|\angle \mathbf{GHA}| = 2|\angle \mathbf{CHA}|$, where the final equation holds because $\angle \mathbf{GHA} = \angle \mathbf{CHA}$. ***<u>This proves the claim.</u>***

Proceeding inductively, we obtain a sequence of points $\mathbf{B}_0, \mathbf{B}_1, \mathbf{B}_2, \ldots$ of points on **(CH** such that $|\angle \mathbf{CB}_{k+1}\mathbf{A}| \leq \tfrac{1}{2}|\angle \mathbf{CB}_k\mathbf{A}|$, and it follows that for each n we have

$$|\angle \mathbf{CB}_n\mathbf{A}| \leq 2^{-n}|\angle \mathbf{CB}_0\mathbf{A}|.$$

If we choose n large enough, we can make the right hand side (hence the left hand side) of this inequality less than $\frac{1}{2}(90° - \theta)$. Furthermore, we can also choose n so that

$$|\angle CB_nA| \; < \; \theta \; = \; |\angle CAW|$$

and it follows that the angle sum for $\triangle AB_nC$ will be

$$|\angle CAB_n| + |\angle AB_nC| + |\angle ACB_n| \; <$$

$$\tfrac{1}{2}(180° - \theta) + \theta + 180° \; < \; (90° - \theta) + \theta + 90° \; = \; 180°.$$

Therefore we have constructed a triangle whose angle sum is less than $180°$, as required.■

Definition. Given $\triangle ABC$ in a hyperbolic plane, its ***angle defect*** is given by

$$\delta(\triangle ABC) \; = \; 180° - |\angle CAB| - |\angle ABC| - |\angle ACB|.$$

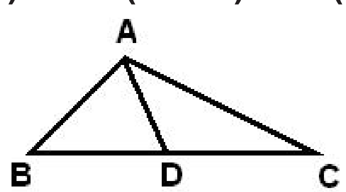
By Theorem 2, ***in a hyperbolic plane the angle defect of $\triangle ABC$ is a positive real number which is always strictly between $0°$ and $180°$.***■

The Hyperbolic Angle – Angle – Angle Congruence Theorem

We have already seen that in spherical geometry there is a complementary notion of ***angle excess***, and the area of a spherical triangle is proportional to its angle excess. There is a similar phenomenon in hyperbolic geometry: ***For any geometrically reasonable theory of area in hyperbolic geometry, the angle of a triangle is proportional to its angular defect.*** This is worked out completely in the book by Moïse. However, for our purposes we only need the following property which suggests that the angle defect behaves like an area function.

Proposition 4. (Additivity property of angle defects) *Suppose that we are given $\triangle ABC$ and that D is a point on (BC). Then we have*

$$\delta(\triangle ABC) \; = \; \delta(\triangle ABD) + \delta(\triangle ADC).$$



Proof. If we add the defects of the triangles we obtain the following equation:

$$\delta(\triangle ABD) + \delta(\triangle ADC) \; = \; 180° - |\angle DAB| - |\angle ABD| - |\angle ADB| +$$
$$180° - |\angle CAD| - |\angle ADC| - |\angle ACD|$$

By the Supplement Postulate for angle measure we know that

$$|\angle ADB| + |\angle ADC| \; = \; 180°$$

by the Additivity Postulate we know that

$$|\angle BAC| = |\angle BAD| + |\angle DAC|$$

and by the hypotheses we also know that $\angle ABD = \angle ABC$ and $\angle ACD = \angle ACB$. If we substitute all these into the right hand side of the equation for the defect sum $\delta(\triangle ABD) + \delta(\triangle ADC)$, we see that this right hand side reduces to

$$180^\circ - |\angle CAB| - |\angle ABC| - |\angle ACB|$$

which is the angle defect for $\triangle ABC$.■

The next result yields a striking conclusion in hyperbolic geometry, which shows that ***the latter does not have a similarity theory comparable to that of Euclidean geometry.***

Theorem 5. (Hyperbolic A.A.A. or *Angle – Angle – Angle Congruence Theorem*) *Suppose we have ordered triples* (A, B, C) *and* (D, E, F) *of noncollinear points such that the triangles* $\triangle ABC$ *and* $\triangle DEF$ *satisfy* $|\angle CAB| = |\angle FDE|$, $|\angle ABC| = |\angle DEF|$, *and* $|\angle ACB| = |\angle DFE|$. *Then we have* $\triangle ABC \cong \triangle DEF$.

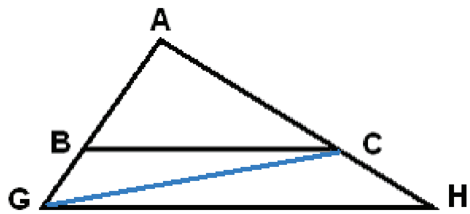
Lecture 16 ends here.

Proof. If at least one of the statements $|BC| = |EF|$, $|AB| = |DE|$, or $|AC| = |DF|$ is true, then by **A.S.A.** we have $\triangle ABC \cong \triangle DEF$. Therefore it is only necessary to consider possible situations in which all three of these statements are false. This means that in each expression, one term is less than the other. There are eight possibilities for the directions of the inequalities, and these are summarized in the table below.

This should be $|DF|$.

CASE	|AB| ?? |DE|	|AC| ?? /DF/	|BC| ?? |EF|
000	$|AB| < |DE|$	$|AC| < |DF|$	$|BC| < |EF|$
001	$|AB| < |DE|$	$|AC| < |DF|$	$|BC| > |EF|$
010	$|AB| < |DE|$	$|AC| > |DF|$	$|BC| < |EF|$
011	$|AB| < |DE|$	$|AC| > |DF|$	$|BC| > |EF|$
100	$|AB| > |DE|$	$|AC| < |DF|$	$|BC| < |EF|$
101	$|AB| > |DE|$	$|AC| < |DF|$	$|BC| > |EF|$
110	$|AB| > |DE|$	$|AC| > |DF|$	$|BC| < |EF|$
111	$|AB| > |DE|$	$|AC| > |DF|$	$|BC| > |EF|$

Reversing the roles of the two triangles if necessary, we may assume that at least two of the sides of $\triangle ABC$ are shorter than the corresponding sides of $\triangle DEF$. Also, if we consistently reorder $\{A, B, C\}$ and $\{D, E, F\}$ in a suitable manner, then we may also arrange things so that $|AB| < |DE|$ and $|AC| < |DF|$. Therefore, if we take points G and H on the respective open rays $(BA$ and $(BC$ such that $|AG| = |DE|$ and $|AH| = |DF|$, then by **S.A.S.** we have $\triangle AGH \cong \triangle DEF$.



By hypothesis and construction we know that the angular defects of these triangles satisfy **δ(△AGH) = δ(△DEF) = δ(△ABC).** We shall now derive a contradiction using the additivity property of angle defects obtained previously. The distance inequalities in the preceding paragraph imply the betweenness statements **A∗B∗G** and **A∗C∗H,** which in turn yield the following defect equations:

$$\delta(\triangle \mathbf{AGH}) = \delta(\triangle \mathbf{AGC}) + \delta(\triangle \mathbf{GCH})$$

$$\delta(\triangle \mathbf{AGC}) = \delta(\triangle \mathbf{ABC}) + \delta(\triangle \mathbf{BGC})$$

If we combine these with previous observations and the positivity of the angle defect we obtain

$$\delta(\triangle \mathbf{ABC}) < \delta(\triangle \mathbf{ABC}) + \delta(\triangle \mathbf{BGC}) + \delta(\triangle \mathbf{GCH}) =$$
$$\delta(\triangle \mathbf{AGH}) = \delta(\triangle \mathbf{DEF})$$

which contradicts the previously established equation **δ(△DEF) = δ(△ABC).** The source of this contradiction is our assumption that the corresponding sides of the two triangles do not have equal lengths, and therefore this assumption must be false. As noted at the start of the proof, this implies **△ABC ≅ △DEF.■**

One immediate consequence of Theorem 6 is that ***in hyperbolic geometry, two triangles cannot be similar in the usual sense unless they are congruent.*** In particular, this means that we cannot magnify or shrink a figure in hyperbolic geometry without distortions. This is disappointing in many respects, but if we remember that angle defects are supposed to behave like area functions then this is not surprising; we expect that two similar but noncongruent figures will have different areas, and in hyperbolic (just as in spherical!) geometry this simply cannot happen.

The Strong Hyperbolic Parallelism Property

The negation of Playfair's Postulate is that there is ***some line*** and ***some external point*** for which ***parallels are not unique.*** It is natural to ask if there are neutral geometries in which unique parallels exist for ***some but not all*** pairs **(L, A)** where **L** is a line and **A** is an external point. The next result implies that no such neutral geometries exist.

Theorem 7. *Suppose we have a neutral plane* $\mathbb{P}$ *such that for* ***some*** *line* **L** *and* ***some external point*** **A** *there is a unique parallel to* **L** *through* **A.** *Then there is a rectangle in* $\mathbb{P}$*.*