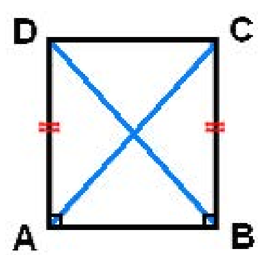
Math 133
Fall 2021

SOLUTIONS FOR WEEK 08 EXERCISES

For these exercises assume that all points lie in a plane which satisfies the axioms for neutral geometry.

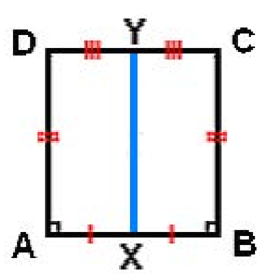
1. Follow the hint. We have $k/h > 0$, so by density of the rationals there is a rational number m/n such that $m, n > 0$ and $0 < m/n < k/h$. But then we also have $0 < 1/n \leq m/n < k/h$, and these inequalities yield $h/n < k$. Since $n < 2^n$ for all positive integers n, it follows that $0 < h/2^n < h/n < k$, as stated in the exercise.■

2. Let L be a line, and take points $A,\ B \in L$ such that $|AB| = q$. Let X be a point which does not lie on L, and consider the plane $\mathcal{P}$ determined by L and X. By the Protractor Postulate there exist points U and V on the same side of L as X (in $\mathcal{P}$) such that $UA \perp AB$ and $VB \perp AB$. The Ruler Postulate then yields points $D \in (AU$ and $C \in (BV$ such that $|AD| = |BC| = p$, and therefore Proposition 3.5 implies that $A,\ B,\ C,\ D$ determine the vertices of a convex quadrilateral; by construction it is a Saccheri quadrilateral.■



3. By SAS we have $\triangle DAB \cong \triangle CBA$, so that the diagonals satisfy $|BD| = |AC|$. Therefore by SSS we have $\triangle CDA \cong \triangle DCB$, and this implies $|\angle CDA| = |\angle DCB|$.■

4. Let X and Y be the midpoints of $[AB]$ and $[CD]$ respectively. Then we have $\triangle DAX \cong \triangle CBX$ by SAS, so that $|XD| = |XC|$, so that XY is the perpendicular bisector of $[CD]$.

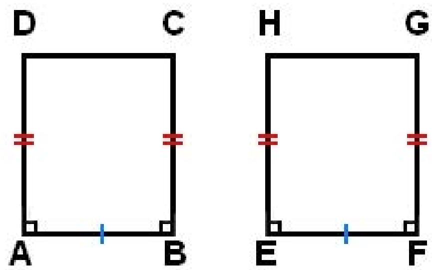


Similarly, we have $\triangle ADY \cong \triangle BCY$ by SAS (this requires the previous exercise!), and therefore $|YA| = |YB|$, so that XY is the perpendicular bisector of $[AB]$. Therefore XY is perpendicular to both AB and CD.■

5. By Exercise 3 it suffices to show that there is a right angle at D (because that result will imply there is also a right angle at C). Since the summit and base have equal length, by SSS we must have $\triangle ADC \cong \triangle CBA$, so that $|\angle ADC| = |\angle CBA| = 90°$.■

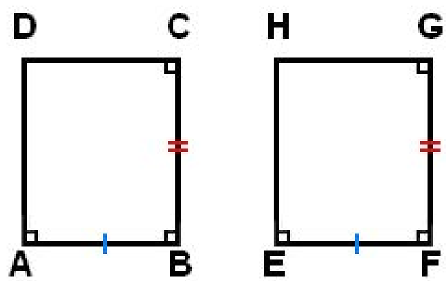
6. The hypotheses imply that $|AB| = |EF|$ and $|AD| = |BC| = |EH| = |FG|$. By **SAS** we have $\Delta DAB \cong \Delta HEF$, and hence we also have $|BD| = |FH|$ and $|\angle DBA| = |\angle HFE|$. Since the $\diamondsuit ABCD$ and $\diamondsuit EFGH$ are Saccheri (hence convex) quadrilaterals, we know that $B \in \mathrm{Int}\, \angle ABC$ and $H \in \mathrm{Int}\, \angle EFG$. By additivity of angle measure, we then obtain

$$|\angle DBC| \quad + \quad 90^\circ \; - \; |\angle DBA| \quad = \quad 90^\circ \; - \; |\angle HFE| \quad = \quad |\angle HFG| \; .$$



Now we can use **SAS** to conclude that $\Delta DBC \cong \Delta HFG$, which implies that $|CD| = |GH|$ — in other words, the summits have equal length — and $|\angle DCB| = |\angle HGF|$. Since the summit angles of a Saccheri quadrilateral have equal measures, it also follows that $|\angle ADC| = |\angle DCB| = |\angle HGF| = |\angle GHE|$, completing the proof.■

7. If we can prove the result with one of the two possible hypotheses on equal lengths, then the other will follow by interchanging the roles of the vertices, so we might as well assume that $|AB| = |EF|$.



By SAS we have $\triangle ABC \cong \triangle EFG$, and hence we also have $|AC| = |EG|$, $|\angle CAB| = |\angle GEF|$, and $|\angle ACB| = |\angle EGF|$. Since a Lambert quadrilateral is automatically a convex quadrilateral, it follows that $C \in \mathrm{Int}\, \angle DAB$ and $G \in \mathrm{Int}\, \angle HEF$; therefore by the additivity of angle measure we have

$$|\angle DAC| \quad + \quad 90^\circ \; - \; |\angle CAB| \quad = \quad 90^\circ \; - \; |\angle GEF| \quad = \quad |\angle GEH| \; .$$

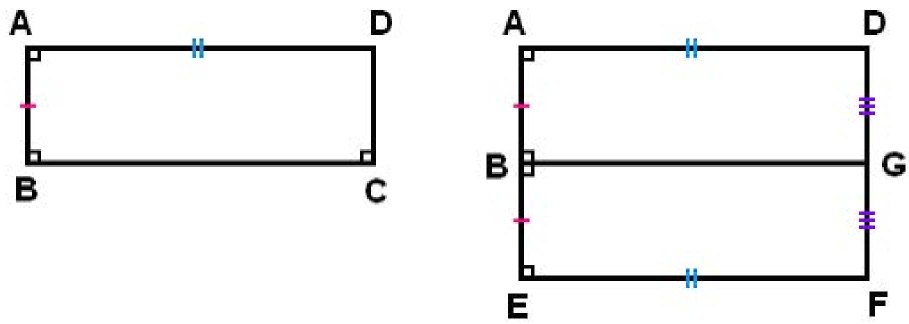
Similarly, we have $A \in \mathrm{Int}\, \angle BCD$ and $E \in \mathrm{Int}\, \angle FGH$, so that

$$|\angle ACD| \quad + \quad 90^\circ \; - \; |\angle ACB| \quad = \quad 90^\circ \; - \; |\angle EGF| \quad = \quad |\angle EGH| \; .$$

Combining these, we see that $\triangle DAC \cong \triangle HEG$ by ASA, so that $|CD| = |GH|$, $|AD| = |EH|$ and $|\angle ADC| = |\angle EHG|$, completing the proof.■

8. Following the hint, we begin by showing that it is enough to show that $|AD| \leq |BC|$. —- If we know this, then we can conclude that $|AB| \leq |CD|$ by reversing the roles of A and C in the discussion which follows.

We know there is a point $E \in (AB$ such that $|AE| = 2 \cdot |AB|$, and since $|AB| < |AE|$ it follows that $A * B * E$. Let $[EX$ be the unique ray such that $EX \perp AB = AE$ and $(EX$ lies on the same side of $AB = AE$ as D, and choose $F \in (EX$ so that $|EF| = |AD|$. Then the points $A,\ E,\ F,\ D$ (in that order) form the vertices of a Saccheri quadrilateral with base $[AE]$.



Let G be the midpoint of $[DF]$. We claim that $G = C$. By Exercise 4 we know that BG is perpendicular to both AB and DF. Since BC is also perpendicular to AB it follows that $BC = BG$. Also, since both CD and GD are perpendicular $BC = BG$ and pass through D , it follows that $CD = GD$. Finally, since CD meets BC in C and GD meets BG in G, it follows that G and C must be the same point.

By the preceding paragraph we have $|DF| = 2 \cdot |CD|$. By Theorem 10.3.4 in Moise (pp. 152–153), we have $|AE| \leq |FD|$, and if we combine these with the defining condition for E we have

$$2 \cdot |AB| \quad = \quad |AE| \quad \leq \quad |DF| \quad = \quad 2 \cdot |CD|$$

and if we divide these inequalities by 2 we obtain the desired relationship $|AB| \leq |CD|$.∎

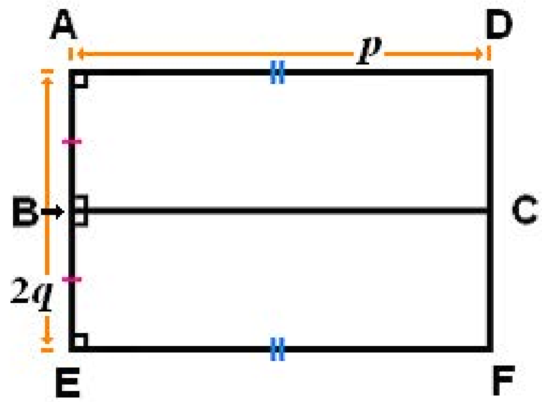
9. As in the preceding exercise, it is enough to prove that the quadrilateral is a rectangle if $|AB| = |CD|$.

It is fairly straightforward to give a proof of this statement which does not involve the construction of the preceding exercise by an argument similar to that for Exercise 7, but there is a very short proof using the Saccheri quadrilateral given above: If we have $|AB| = |CD|$, then it follows that

$$|AE| \quad = \quad 2 \cdot |AB| \quad = \quad 2 \cdot |CD| \quad = \quad |DF|$$

and hence the auxiliary Saccheri quadrilateral is a rectangle. But this means that $\angle ADC = \angle ADF$ is a right angle, which in turn implies that the original Lambert quadrilateral is also a rectangle.∎

10. By Exercise 2 we know that there is a Saccheri quadrilateral with vertices $A,\ E,\ F,\ D$ (in that order) and base $[AE]$ such that $|AE| = 2q$ and $|AD| = p$.



If B and C are the midpoints of $[AE]$ and $[DF]$ respectively, then we know that BC is perpendicular to both AE and DF, and hence the points $A,\ B,\ C,\ D$ form the vertices of a Lambert quadrilateral with right angles at $A,\ B,\ C$. By construction we have $|AD| = p$ and $|AB| = \frac{1}{2} \cdot |AE| = q$.∎