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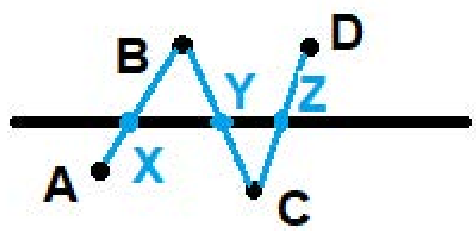
Mathematics 133, Spring 2022, Examination 1

Answer Key

1. [*25 points*] Suppose we are given a line L in a plane P, and A, B, C, D are distinct points of $P - L$ such that $A * X * B$, $B * Y * C$ and $C * Z * D$ hold for some points $X, Y, Z \in L$. Determine which of the points B, C, D lie on the same side of L (in P) as A, and give reasons for your answer.

SOLUTION

Here is a drawing for the problem:

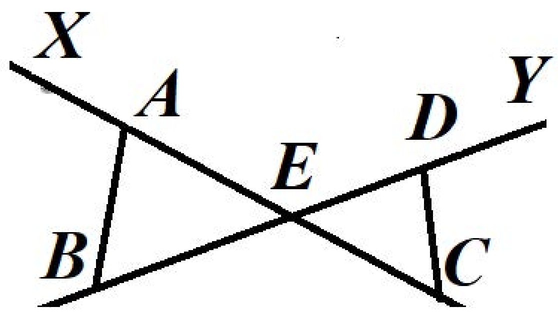


Since neither A nor B lies on L and $X \in (AB) \cap L$, it follows that (i) the points A and B lie on opposite sides of L. Furthermmore, since $Y \in (BC) \cap L$ and $Z \in (CD) \cap L$, it follows that (ii) the points B and C lie on opposite sides of L and (iii) the points C and D lie on opposite sides of L. By (i) and (ii) we know that A and C both lie on the opposite side of L as B, and by (ii) and (iii) we know that B and D both lie on the opposite side of L as C. Therefore B and D lie on the opposite side of L as A. **To summarize,** C is the only point which lies on the same side of L as A.∎

2. [*25 points*] Suppose that we are given lines AC and BD which meet at a point $E \in (AC) \cap (BD)$. Let X and Y satisfy $X * A * E$ and $Y * D * E$ respectively. Prove that $|\angle XAB| > |\angle DEC|$.

SOLUTION

Here is a drawing for the problem:

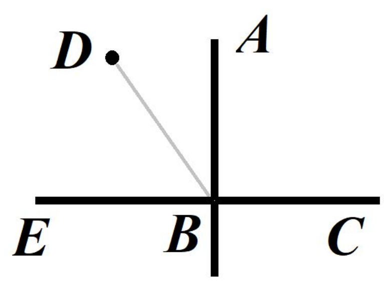


By the Vertical Angle Theorem, $|\angle AEB| = |\angle CED|$. Furthermore, by the Exterior Angle Theorem $|\angle XAB| > |\angle AEB|$. Combining these, we conclude that $|\angle XAB| > |\angle CED|$.■

3. [*25 points*] Suppose that lines AB and BC are perpendicular, and suppose also that D is a point on the same side (= half-plane) of BC as A such that $D \notin$ Interior $\angle ABC$ and $D \notin AB$. Determine which of the statements $|\angle DBA| > 90°$ or $|\angle DBA| < 90°$ is true and give reasons for your answer.

SOLUTION

Let E be a point such that $C * B * E$. Since D and A lie on the same side of BC as A but neither $D \notin$ Interior $\angle ABC$ nor $D \notin AB$, it follows that D and C lie on the same side of AB, it follows that $D \in$ Interior $\angle ABE$ (this was proved in the lectures).

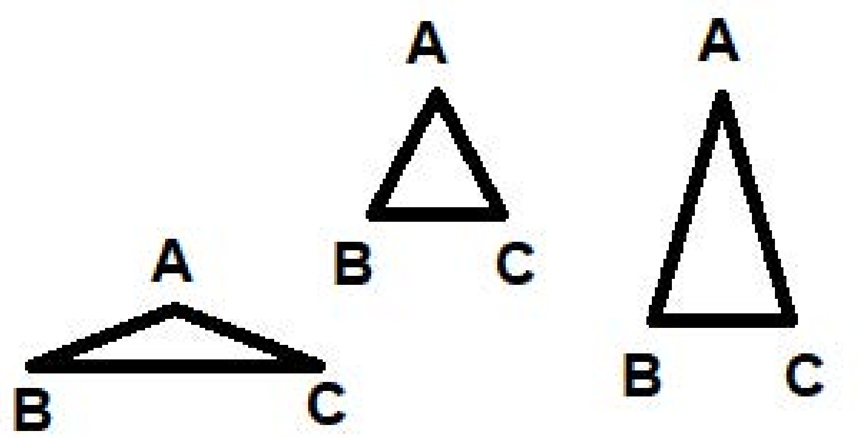


By the Addition Postulate for angle measurement we have $|\angle DBA| < |\angle ABE|$. Also, by the Supplement Postulate and $AB \perp BE(= BC)$, it follows that $|\angle ABE| = 90°$, and therefore we have $|\angle DBA| < 90°$.∎

4. [*25 points*] Assume the plane under consideration is Euclidean, and suppose that we are given isosceles $\triangle ABC$ with $|AB| = |AC|$. Prove that $|AB| > |BC|$ if $|\angle BAC| < 60^\circ$ and $|AB| < |BC|$ if $|\angle BAC| > 60^\circ$.

SOLUTION

Here are some drawings in which $|\angle BAC|$ varies:



Since $\triangle ABC$ is isosceles, we have $|\angle ABC| = |\angle ACB|$, and since we are assuming the Euclidean Parallel Postulate we also have

$$180^\circ \;=\; |\angle ABC| \;+\; |\angle ACB| \;+\; |\angle ACB| \;=\; 2 \cdot |\angle ABC| \;+\; |\angle ACB| \;.$$

Since $|AB| = |AC|$, by the Isosceles Triangle Theorem we have $|\angle ABC| = |\angle ACB|$, so these equations may be rewritten in the form

$$2 \cdot |\angle ACB| \;=\; 180^\circ \;-\; |\angle BAC| \;.$$

If $|\angle BAC| < 60^\circ$ then we must have $|\angle ACB| > 60^\circ$, and since the longer side is opposite the larger angle it follows that $|AB| > |BC|$.

On the other hand, if $|\angle BAC| > 60^\circ$ then we must have $|\angle ACB| < 60^\circ$, and since the longer side is opposite the larger angle it follows that $|AB| < |BC|$.■