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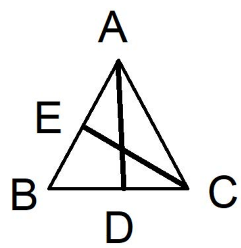
Mathematics 133, Spring 2022, Examination 2

Answer Key

1. [*25 points*] Suppose that we are given $\triangle ABC$ in a **neutral plane** with $D \in (BC)$ and $E \in (AB)$ such that $[AD$ bisects $\angle BAC$ and E is the midpoint of (AB). Prove that (AD) and (CE) have a point in common.

SOLUTION

Here is a drawing for the problem:



By the Crossbar Theorem there is a point $F \in (AD \cap (CE)$, which further implies that $F \in \mathrm{Int}\,\angle BAC$ (the latter is equal to $\angle EAC$); to complete the proof we must show that $F \in (AD)$.

Note first that $F \neq D$ because $F \in CE$ and $CE \cap AC = \{C\}$; then $F = D$ would imply $F \in CE \cap BC = \{C\}$, which contradicts $F \in \mathrm{Int}\,\angle BAC$. The betweenness relations $C * F * E$ and $A * E * B$ imply that A, E, and F all lie on the same side of BC. Since $F \in (AD$ and $AD \cap BC = \{D\}$, it follows that $A * D * F$ does not hold. The only remaining possibility is that $A * D * F$, or equivalently $F \in (AD)$.■

2. [*20 points*] Find the center of the circle which passes through all three vertices of $\triangle ABC$, where $A = (0,0)$, $B = (1,2)$ and $C = (4,3)$.

SOLUTION

Let (u,v) denote the center of this circle, and assume the radius is r. Then we have the following three equations:

$$u^2 + v^2 = r^2 \quad , \quad (u-1)^2 + (v-2)^2 = r^2 \quad , \quad (u-4)^2 + (v-3)^2 = r^2$$

If we subtract the first equation from the second and third equations, we see that

$$(1-2u) + (4-4v) = 0 = (16-8u) + (9-6v)$$

which further simplifies to the system

$$5 = 2u + 4v \quad , \qquad 25 = 8u + 6v \ .$$

Solving for u and v, we find that $u = 7/2$ and $v = -1/2$.■

3. [*30 points*] (a) If a, b, c, d are positive real numbers such that $a/b = c/d$, prove that

$$\frac{a+b}{b} \quad = \quad \frac{c+d}{d} \quad \text{and} \quad \frac{a}{a+b} \quad = \quad \frac{c}{c+d} \ .$$

[*Hints:* What happens if we add 1 to both sides of the original equation, and why is the original equation equivalent to $b/a = d/c$?]

(b) Suppose that we are given points in a **Euclidean plane** such that $A * B * C$ and $D * E * F$ with $|AB|/|BC| = |DE|/|EF|$. Prove that $|AB|/|AC| = |DE|/|DF|$.

SOLUTION

(a) We begin by verifying the first identity.

$$\frac{a}{b} \quad = \quad \frac{c}{d} \quad \Longrightarrow \quad \frac{a}{b} \ + \ 1 \quad = \quad \frac{c}{d} \ + \ 1$$

$$\frac{a+b}{b} \quad = \quad \frac{a}{b} \ + \ 1 \quad = \quad \frac{c}{d} \ + \ 1 \ = \quad \frac{c+d}{d}$$

Changing variables, we also obtain the identity

$$\frac{a+b}{a} \quad = \quad \frac{c+d}{c}$$

and the second identity is obtained by taking the reciprocals of both sides.∎

(b) The betweenness relations imply that $|AC| = |AB|+|BC|$ and $|DF| = |DE|+|EF|$. If we substitute these into the second identity we obtain the equations

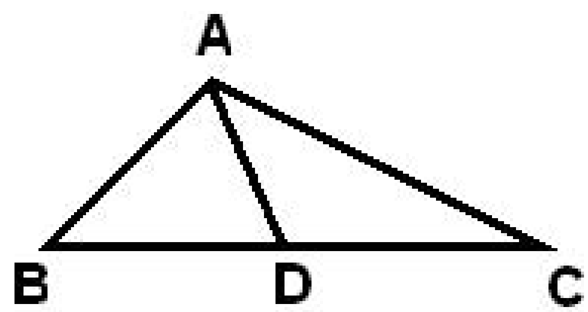
$$\frac{|AB|}{|AC|} \quad = \quad \frac{|AB|}{|AB| + |BC|} \quad = \quad \frac{|DE|}{|DE| + |EF|} \quad = \quad \frac{|DE|}{|DF|}$$

which yield the second identity.∎

4. [*25 points*] Suppose that we are given $\triangle ABC$ in a **hyperbolic plane** with $D \in (BC)$, $E \in (AC)$ and $F \in (AB)$. Prove that the angular defects $\delta\triangle ABC$ and $\delta\triangle ABD$ satisfy $\delta\triangle ABD < \delta\triangle ABC$.

SOLUTION

Here is a drawing:



Since the angular defect is always positive in hyperbolic geometry, it suffices to show that $\delta\triangle ABC = \delta\triangle ABD + \delta\triangle ADC$. By definition, the right hand side is equal to the following sum:

$$(180^\circ - |\angle BAD| - |\angle ABD| - |\angle ADB|) \ + \ (180^\circ - |\angle CAD| - |\angle ACD| - |\angle ADC|)$$

By the Additivity Postulate for angle measures we have $|\angle BAC| = |\angle BAD| + |\angle ADC|$, and by the Supplement Postulate we also have $|\angle ADB| + |\angle ADC| = 180^\circ$. Therefore the sum of the angle defects is equal to

$$360^\circ \ - \ |\angle BAC| \ - \ 180^\circ \ - \ |\angle ABD = \angle ABC| \ - \ |\angle ACD = \angle ACB|$$

which simplifies to the defining equation for $\delta\triangle ABC$ by using the substitution $180 = 360 - 180$.■

5. [*25 points*] For each statement below, state whether it is true in every neutral plane, only in a Euclidean plane, or only in a hyperbolic plane. Correct answers do not need supporting reasons, but partial credict may be given for incorrect answers if brief, substantial reasons are included. **NOTE:** It is also possible that a statement is false in **every** neutral plane, and this should be included in the possible responses for each item.

(a) Two lines perpendicular to a third line are parallel.

(b) If two lines are parallel and one of them is perpendicular to a third line, then so is the other.

(c) The hypotenuse-side congruence theorem for right triangles.

(d) The SSA congruence theorem for arbitrary triangles.

(e) The Exterior Angle Theorem.

(f) The AAA congruence theorem for triangles.

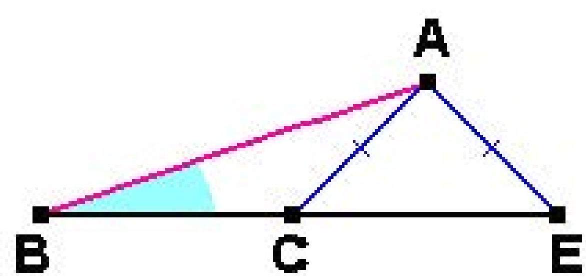
SOLUTION

(a) Always true in neutral geometry.

(b) Only true in Euclidean geometry.

(c) Always true in neutral geometry.

(d) Always false in neutral geometry. — Here is a drawing:



(e) Always true in neutral geometry.

(f) Only true in hyperbolic geometry.