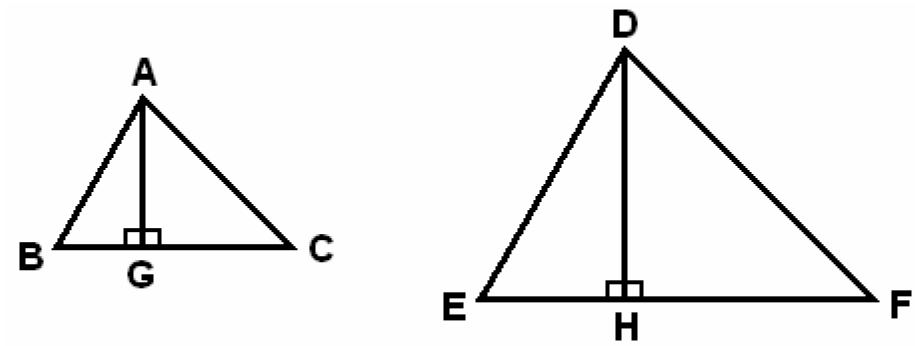
Comments on two problems from

Mathematics 133, Winter 2009, Examination 3

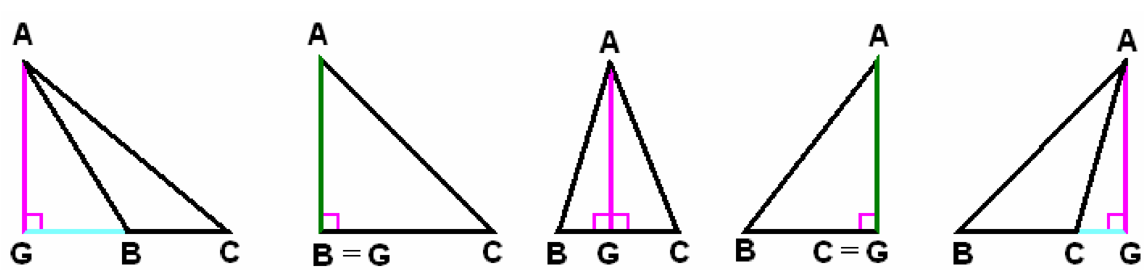
2. Here is a drawing for the problem:



Further discussion. The conclusion of the exercise remains valid regardless of where the feet of the perpendiculars lie, but not surprisingly this takes additional work. For the sake of completeness, we shall describe how one can prove the more general result using material introduced up to and including Section **III.5.**

Before doing so, we note that the general conclusion follows very quickly if we use the formula for the altitude derived in the proof of Heron's formula (see the proof of Theorem **III.7.6**). Alternatively, one can use the familiar formula ***Area* = ½ (*base*)(*height*)** and the fact (not shown in the notes) that if two triangles are similar with ratio of similitude k, then the ratio of the areas for the closed regions bounded by these triangles is equal to k^2.

The discussion splits into five cases, depending upon where the foot **G** of the perpendicular from **A** to **BC** is situated with respect to **B** and **C.** As indicated in the drawing below, the possibilities are **G*B*C, G = B, B*G*C, G = C,** and **B*C*G.**



Of course, if we are given another triangle △**DEF** and **H** is the foot of the perpendicular from **D** to **EF,** there are five analogous possibilities, and a crucial step is to show that for each possible case for △**ABC** there is a unique corresponding case for △**DEF.** Here is a formal statement of what is true:

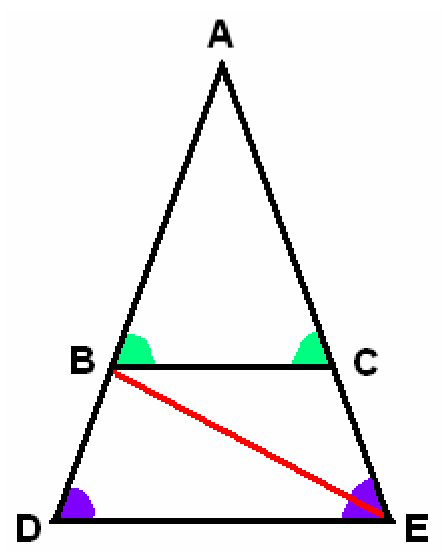
CLAIM. *Suppose that* △**ABC** ~ △**DEF,** *and let* **G** *and* **H** *be the feet of the perpendiculars from* **A** *to* **BC** *and from* **D** to **EF.** *Then we have the following*:

1. *If* **G∗B∗C** *is true, then* **H∗E∗F** *is true.*
2. *If* **G = B,** *then* **H = E.**
3. *If* **B∗G∗C** *is true, then* **E∗H∗F** *is true.*
4. *If* **G = C,** *then* **H = F.**
5. *If* **B∗C∗G** *is true, then* **E∗F∗H** *is true.*

One can show this by applying the Exterior Angle Theorem systematically. The first possibility occurs if and only if ∠**ABC** is obtuse, the second if and only if ∠**ABC** is a right angle, the third if and only if both ∠**ABC** and ∠**ACB** are acute, the fourth if and only if ∠**ACB** is a right angle, and the fifth if and only if ∠**ACB** is obtuse. The similarity of the two triangles implies that |∠**ABC**| = |∠**DEF**| and |∠**ACB**| = |∠**DFE**|, and thus in each of the cases the ordering relationship among **B, C** and **G** yields the corresponding ordering relationship among **D, E** and **H.**

In particular, the third case is assumed to hold in the exercise, and we see that it is enough to assume just one of the ordering relationships because the other can be derived from it as in the preceding discussion. Note also that in the second and fourth cases the conclusion of the exercise follows immediately from the definition of similar triangles. Finally, we note that the fifth case can be handled by the same argument which we employed to dispose of the third case, and we can retrieve the first case from the fifth case by interchanging the roles of **B** and **C** and of **E** and **F** in the argument.

4. Here is a drawing for this problem:



In this picture, the base angles of the isosceles triangles △**ABC** and △**ADE** are marked in green and purple respectively; by the Isosceles Triangle Theorem, the measures of the base angles marked with the same color are equal. Repeated applications of Proposition **V.4.4** in the notes imply that

δ(△ADE) = δ(△ABE) + δ(△EBD), δ(△ABE) = δ(△EBD) + δ(△ABC)

which in turn leads to the inequalities

$$\delta(\triangle \mathbf{ABC}) \;<\; \delta(\triangle \mathbf{ABE}) \;<\; \delta(\triangle \mathbf{ADE})$$

and as indicated in the answer key for the examination this chain of inequalities implies that $|\angle \mathbf{ADE}| < |\angle \mathbf{ABC}|$.

Of course, the situation in Euclidean geometry is entirely different, for in that case we know that $\triangle \mathbf{BAC} \sim \triangle \mathbf{DAE}$ by the **SAS** Similarity Theorem, and therefore we have $|\angle \mathbf{ADE}| = |\angle \mathbf{ABC}|$.