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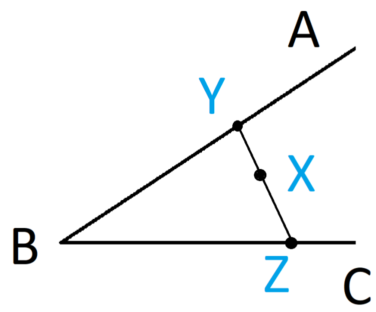
Mathematics 133, Fall 2020, Examination 2

Answer Key

1. [*25 points*] Assume we are working in the coordinate plane. Let $\angle ABC$ be given, let X be a point in the interior of $\angle ABC$, and let $Y \in (BA$. Assume also that the line XY meets $(BC$ at a point Z. Which of the three points X, Y, Z is between the other two? Give reasons for your answer.

SOLUTION

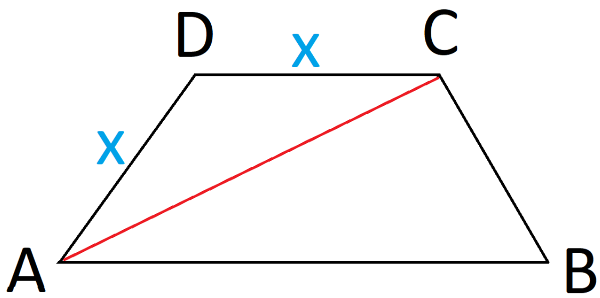


The drawing indicates that we should have $Y * X * Z$. To verify this, we need the following observation from Chapter II: *Let L be a line containing the collinear points P, Q and R, and let M be a second line passing through Q. Then P and R are on the same side of M if and only if Q is not between P and R.* Proof: Since the two open half-planes defined by M are convex, if $P * Q * R$ is true and P, R lie on the same side of M, then Q also lies on this half-plane, contradicting our assumption that $Q \in M$; therefore $P * Q * R$ implies that P and R lie on opposite sides of M. Conversely, if P and R lie on opposite sides of M, then there is some point $X \in (PR) \cap M$. We already know that $Q \in M$, and since two lines only have one point in common it follows that $Q = X$ and hence $P * Q * R$.■

We now apply this to the given situation. Since X lies in the interior of $\angle ABC$, we know that it lies on the same side of AB as C and Z, and also on the same side of BC as A and Y (note that $\angle ABC = \angle YBZ$). By the preceding paragraph, the first of these eliminates the possibility $Z * Y * X$, and the second eliminates the possibility $X * Z * Y$. Since one of the three points X, Y, Z is between the other two, the only remaining possibility is $Y * X * Z$.■

2. [*25 points*] Suppose that we are working in a Euclidean plane $\mathbb{P}$, and let $ABCD$ denote a (convex) trapezoid with $AB||CD$. Assume further that $|AD| = |DC|$. Prove that $[AC$ bisects $\angle DAB$.

SOLUTION



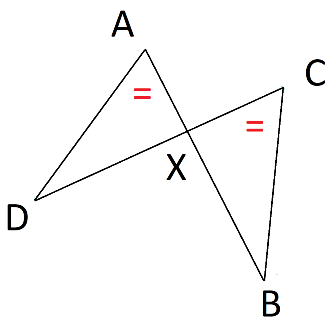
By the Isosceles Triangle Theorem we have $|\angle DCA| = |\angle DAC|$. Since we know that $ABCD$ is a convex quadrilateral, the result on intersections of its diagonals implies that D and B are on opposite sides of AC. Therefore $\angle DCA$ and $\angle CAB$ are alternate interior angles, and hence the given condition $AB||CD$ implies that $|\angle DCA| = |\angle CAB|$. Finally, since the convexity of $ABCD$ implies that C lies in the interior of $\angle DAB$ and therefore we have

$$|\angle DAB| \quad = \quad |\angle DAC| + |\angle CAB| \quad = \quad |\angle DCA| + |\angle CAB| \quad = \quad 2 \cdot |\angle CAB|$$

if we use the two angle measure equations established in previous steps. These equations show that $[CA$ bisects $\angle DAB$.■

3. [*25 points*] Assume that we are given two angles $\angle DAB$ and $\angle DCB$ in a Euclidean plane $\mathbb{P}$, and suppose that we have $X \in (AB) \cap (CD)$. Prove the equation $|AX| \cdot |XB| \ = \ |CX| \cdot |XD|$. **Assume also that** **$|\angle DAB| = |\angle DCB|$.**

SOLUTION



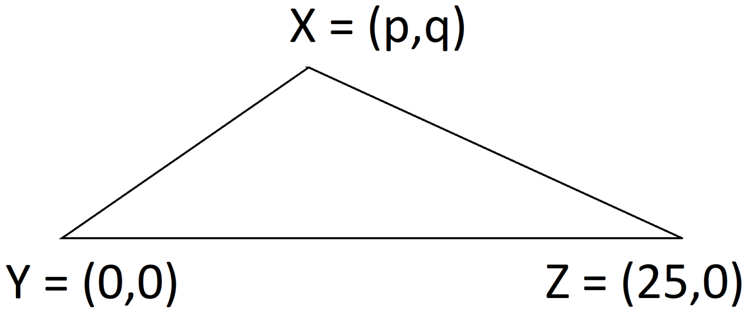
By the Vertical Angle Theorem we know that $|\angle AXD| = |\angle CXB$, and hence by the AA Similarity Theorem we have $\triangle AXD \sim \triangle CXB$. The latter yields the proportionality equation

$$\frac{|AX|}{|CX|} \quad = \quad \frac{|DX|}{|BX|}$$

and if we clear this of fractions we find that $|AX| \cdot |XB| \ = \ |CX| \cdot |XD|$.■

4. [*25 points*] As in Quiz 2, take the last four digits $ABCD$ of your student identification number, and once again consider the point in the coordinate plane given by $X = (A+B, C+D)$; let $Y = (0,0)$ and $Z = (25,0)$. Find the orthocenter of $\triangle XYZ$. The proof of the theorem on orthocenters yields one way of solving this problem.

SOLUTION



To simplify the notation let $p = A + B$ and $q = C + D$ as in the drawing. By the concurrence of the altitudes it suffices to find the point where the altitudes from X and Z meet. Since YZ is horizontal, the altitude from X to YZ is a vertical line and since $X = (p, q)$ this line must have equation $x = p$. The slope of the perpendicular from Z to XY is the negative reciprocal of the slope of XY; since the slope of the latter line is q/p, it follows that the slope of the perpendicular from Z is $-p/q$. Therefore this perpendicular line has an equation of the form

$$y = c - \frac{px}{q}$$

for some constant c; since $Z = (25, 0)$ lies on this line the constant c satisfies

$$0 = c - \frac{25p}{q} \qquad \text{so that} \qquad c = \frac{25p}{q} .$$

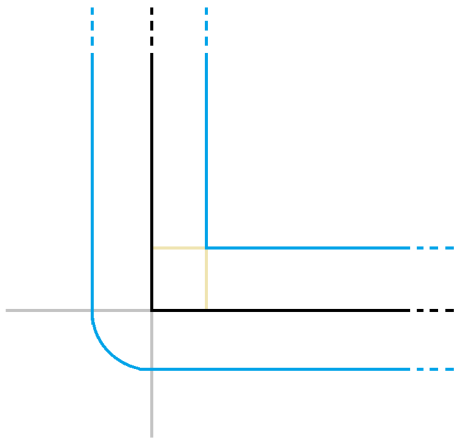
Therefore we can write the equation of this perpendicular as $y = p(25-x)/q$. Now the first line's equation is $x = p$ and if we combine everything we see that the two perpendiculars meet at the point

$$\left(p, \frac{p(25-p)}{q} \right)$$

where p and q are given as above in terms of A, B, C, D.■

5. [*25 points*] Let A be the set of all points in the coordinate plane $\mathbb{R}^2$ which are on either the nonnegative x-axis or the nonnegative y-axis (hence A = all points of the form (u,v) where either $u \geq 0$ and $v = 0$ or else $u = 0$ and $v \geq 0$). Describe the set L of all points (p,q) such that the (shortest) distance from (p,q) to L is equal to 1. Describe the points of L in numerical terms (equations and inequalities involving the coordinates p and q). There are four cases corresponding to the four quadrants of the coordinate plane.

SOLUTION



In the picture above, the sets A and L are drawn in black and blue respectively, and the remaining parts of the coordinate axes are drawn in gray. Let A_1 and A_2 denote the rays determined by the nonnegative x and y coordinate axes respectively. Note that the minimum distance from a point X to A is the smaller of the minimum distance to A_1 and the minimum distance to A_2. We need to give a complete description for the points of L in each of the four closed quadrants in the coordinate plane.

FIRST CLOSED QUADRANT. *In this case* $x, y \geq 0$. As above. the minimum distance from a point X to A is the smaller of the minimum distance to A_1 and the minimum distance to A_2. Since the shortest distance from a point to a line is along a perpendicular, these minima are x and y respectively. So the set of all first quadrant points is all (x,y) so that the minimum of x and y is equal to 1. In other words, The portion of A in the first quadrant is all points (x,y) in that quadrant such that $y \geq x \geq 1$ or $x \geq y \geq 1$, which is the union of the two rays $\{1\} \times [1,\infty)$ and $[1,\infty) \times \{1\}$.■

SECOND CLOSED QUADRANT. *In this case* $y \geq 0 \geq x$. Once again, since the shortest distance from a point to the line of A_2 is a common perpendicular, it follows that the minimum distance from a point (x,y) in the second quadrant to a point of A_2 is equal to $|x|$. Now consider the distance from (x,y) to a point $(z,0) \in A_1$ where $z \geq 0$; this distance is equal to

$$\sqrt{(x-z)^2 + y^2} \quad = \quad \sqrt{(|x|+z)^2 + y^2}$$

because $z \geq 0 \geq x$. The right hand side is greater than or equal to $|x|$ with equality if and only if $y = z = 0$. Therefore the distance from (x,y) to A is equal to $|x|$, and in particular it is equal to 1 if and only if $x = -1$.■

THIRD CLOSED QUADRANT. *In this case* $x, y \leq 0$. Since $(0,0) \in A$ it follows that the minimum distance from (x, y) to A is at least $\sqrt{x^2 + y^2}$. The picture suggests that if the latter is 1, then the minimum distance from (x, y) to A is exactly 1. More generally, we claim that the minimum distance from (x, y) to A is exactly $\sqrt{x^2 + y^2}$. Every point of A has the form (a, b) where $a, b \geq 0$ and at least one coordinate is zero. The distance from (x, y) to (a, b) is equal to

$$\sqrt{(x-a)^2 + (y-b)^2} \quad = \quad \sqrt{(|x|+a)^2 + (|x|+b)^2}$$

because $a, b \geq 0 \geq x, y$. The right hand side is greater than or equal to $\sqrt{x^2 + y^2}$ with equality if and only if $a = b = 0$. Thus the distance from (x, y) to A is exactly $\sqrt{x^2 + y^2}$, and therefore the intersection of A with the third quadrant is equal to the portion of the circle $x^2 + y^2 = 1$ within that quadrant.■

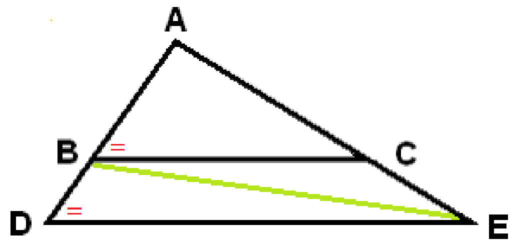
FOURTH CLOSED QUADRANT. *In this case* $x \geq 0 \geq y$. The argument in this case is basically the same as for the second closed quadrant with the roles of the coordinates reversed. The shortest distance from a point to the line of A_1 is a common perpendicular, it follows that the minimum distance from a point (x, y) in the second quadrant to a point of A_2 is equal to $|y|$. Now consider the distance from (x, y) to a point $(0, z) \in A_2$ where $z \geq 0$; this distance is equal to

$$\sqrt{x^2 + (y-z)^2} \quad = \quad \sqrt{x^2 + (|y|+z)^2}$$

because $z \geq 0 \geq y$. The right hand side is greater than or equal to $|y|$ with equality if and only if $x = z = 0$. Therefore the distance from (x, y) to A is equal to $|y|$, and in particular it is equal to 1 if and only if $y = -1$.■

6. [*25 points*] Assume that all points arising in this discussion lie in a hyperbolic plane $\mathbb{P}$. Suppose that we are given $\triangle ADE$ with $B \in (AD)$ and $C \in (AE)$ such that $|\angle ABC| = |\angle ADE|$. Is $|\angle ACB|$ greater than, equal to or less than $|\angle AED|$? Prove that your answer is correct.

SOLUTION



We shall use apply the angle defect function for the hyperbolic triangles under consideration. This yields the equations

$$\delta\triangle ADE \;=\; \delta\triangle ABE + \delta\triangle BDE \;=\; \delta\triangle ABC + \delta\triangle BCE + \delta\triangle BDE \;>\; \delta\triangle ABC$$

because the defect of a hyperbolic triangle is always positive. The inequality may then be rewritten as follows:

$$180 - |\angle DAE| - |\angle ADE| - |\angle AED| \;=\; \delta\triangle ADE \;>$$

$$\delta\triangle ABC \;=\; 180 - |\angle BAC| - |\angle ABC| - |\angle ACB|$$

Since $\angle DAE = \angle BAC$ and $|\angle ADE| = |\angle ABC|$ the inequality in the displayed expression reduces to $-|\angle AED| > -|\angle ACB|$, which is equivalent to $|\angle ACB| > |\angle AED|$.■