Mathematics 133, Fall 2021, Examination 2

Answer Key

1. [*25 points*] Assume we are working in neutral plane geometry, and let AB be a line in that plane with $X \in [AB$. If $|AX| < |AB|$ which of the following are true and which are false? Reasons are not required but might earn partial credit if your answer is incorrect.

(a) $B \in (AX)$.

(b) $[AB = [AX$.

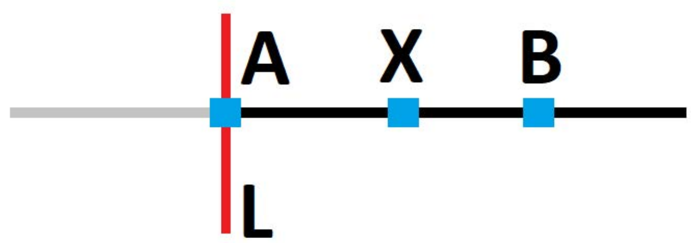
(c) If L is a second line through A, then B and X lie on opposite sides of L.

(d) If $C \notin AB$ then X lies in the interior of $\angle ACB$.

(e) $X \in [BA$.

SOLUTION

We shall start by taking a ruler function $f : AB \to \mathbb{R}$ such that $f(A) = 0$ and $f(B) > 0$. Then $X \in [AB$ implies that $f(X) > 0$ and hence $f(X) = |f(X) - f(A)| = |AX| < |AB| = |f(B) - f(A)| = f(B)$.



(a) **FALSE.** If this were true, then $A * B * X$ would imply $|AB| < |AX|$, which contradicts the assumption that $|AX| < |AB|$.■

(b) **TRUE.** We have noted that $0 < f(X) < f(A)$, and we know that the rays $[AB$ and $[AX$ are the sets of points Y on the line such that $f(Y) \geq 0$ in each case.■

(c) **FALSE.** If this were true, then by the Plane Separation Postulate there would be a point $Y \in (BX) \cap L \subset AB \cap L$. Since $A \in AB \cap L$ and two distinct lines have at most one point in common, it follows that $A = Y$ and hence $B * A * X$. This contradicts the defining condition for X to lie on $[AB$, and therefore B and X cannot lie on opposite sides of L.■

(d) **FALSE.** By definition the interior of $\angle ACB$ is contained in one of the half-planes defined by AB, and therefore the subsets AB and $\text{Int}\,\angle ACB$ have no points in common. Since $X \in AB$ this means that $X \notin \text{Int}\,\angle ACB$.■

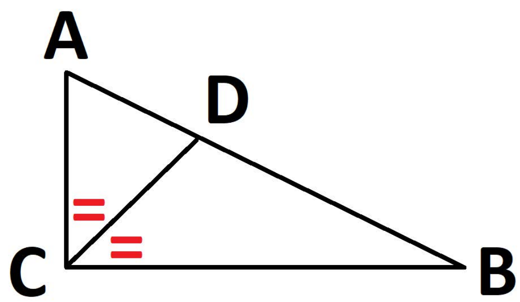
(e) **TRUE.** Since $X \in [AB$ and $|AX| < |AB|$ we know that $X \in (AB)$.■

2. [*20 points*] Assume we are working in Euclidean plane geometry and we are given a right triangle $\triangle ABC$ with a right angle at C, and let $D \in (AB)$ be a point such that $[CD$ bisects $\angle ACB$. Give a formula for $|AD|$ in terms of $|AB|$, $|BC|$ and $|AC|$.

SOLUTION

By the Angle Bisector Theorem in Euclidean geometry, we know that

$$\frac{|AC|}{|BC|} \quad = \quad \frac{|AD|}{|BD|} \quad = \quad \frac{|AD|}{|AB| - |AD|}$$



and if we let $x = |AD|$ this equation reduces to

$$\frac{|AC|}{|BC|} \quad = \quad \frac{x}{|AB| - x}$$

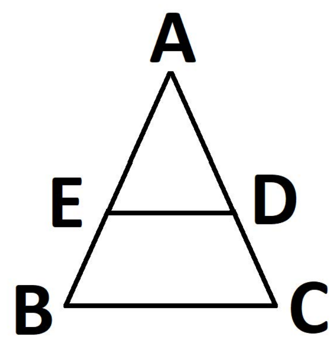
which in turn is equivalent to $|AC| \cdot (|AB| - x) \;=\; x \cdot |BC|$. If we solve for $x = |AD|$ we find that is equal to

$$\frac{|AC| \cdot |AB|}{|AC| + |BC|} \;. \blacksquare$$

3. [*30 points*] Assume we are working in Euclidean plane geometry and we are given $\triangle ABC$, with $D \in (AC)$ and $E \in (AB)$ such that BC is parallel to DE. Prove that $|BE| = |CD|$ if and only if $|\angle EBC| = |\angle DCB|$.

SOLUTION

Here is a drawing which reflects the first sentence of the problem. However, one must observe that the given data contain no assmption whether the triangles are isosceles.



What can we conclude? Since $BC||DE$ the theorem on transversals and corresponding angles implies that $|\angle ABC| = |\angle ADE|$ and $|\angle ACB| = |\angle AED|$, so that $\triangle ABC \sim \triangle ADE$ by the AA Similarity Theorem. Therefore

$$\frac{|AB|}{|AE|} = \frac{|AC|}{|AD|} \quad \text{or eqivalently} \quad \frac{|AB|}{|AC|} = \frac{|AE|}{|AD|} .$$

The betweenness relations imply that $|AB| = |AE| + |BE|$ and $|AC| = |AD| + |CD|$. If we substitute this into the left hand side of the first proportionality equation, we obtain yet another equivalent version

$$1 + \frac{|BE|}{|AE|} = 1 + \frac{|CD|}{|AD|}$$

which in turn is equivalent to $|BE|/|AE| = |CD|/|AD|$ and $|BE|/|CD| = |AE|/|AD|$.

Suppose now that $|BE| = |CD|$. Then the left hand side of the last equation is equal to 1, and thus the same is true for the right hand side, so that $|AE| = |AD|$. This means that the ratio on the right hand side of the second equation is also equal to 1, so that $|AB|/|AC| = 1$. By the Isosceles Triangle Theorem we then have $|\angle ABC = \angle EBC| = |\angle ACB = \angle DCB|$, proving the implication in one direction.

Convrsely, suppose that $|\angle EBC| = |\angle DCB|$. By the last sentence of the previous paragraph this can be rewritten as $|\angle ABC| = |\angle ACB|$. The Isosceles Triangle Theorem now implies that $|AB| = |AC|$ and hence $|AB|/|AC| = 1$. Since $|AB|/|AC| = |AE|/|AD|$ it also follows that $|AD| = |AE|$ If we substitute the equations of this paragraph into the expressions $|AB| = |AE| + |BE|$ and $|AC| = |AD| + |CD|$, we have

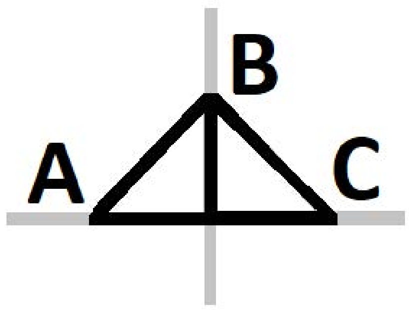
$$|AE| + |BE| = |AB| = |AC| = |AC| = |AD| + |CD| = |AE| + |CD|$$

and if we subtract $|AE|$ from the left and right hand sides we obtain the desired equation $|BE| = |CD|$.∎

4. [*25 points*] Assume we are working in the coordinate plane $\mathbb{R}^2$ and we are given the isosceles right triangle $\triangle ABC$ where $A = (-1, 0)$, $B = (0, 1)$ and $C = (1, 0)$. Prove that there are two points D on the y-axis such that $2\,|BD| = |AD| = |CD|$ and find the y-coordinates for these points.

SOLUTION

It is helpful to plot the given points in the coordinate plane.



The equation $2\,|BD| = |AD| = |CD|$ is equivalent to $4\,|BD|^2 = |AD|^2 = |CD|^2$, and it will be convenient to work with this formulation. If $D = (0, y)$ the latter equation can be rewritten in the form

$$4(y-1)^2 \quad = \quad y^2 + 1$$

which is equivalent to $3y^2 - 8y + 3 \ = \ 0$. If we solve for y using the Quadratic Formula, we obtain the following two roots:

$$y \quad = \quad 4 \ \pm \ \sqrt{16 - 9 = 7}$$

Therefore the two choices for D are $\left(0, 4 \pm \sqrt{7}\right)$.■

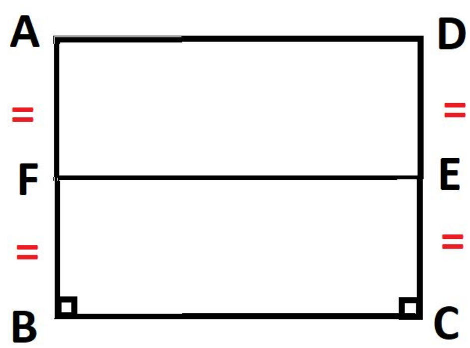
5. [*30 points*] Assume we are working in a neutral plane geometry and we are given a Saccheri quadrilateral $\diamondsuit ABCD$ with right angles at B and C, and let E and F be the midpoints of $[CD]$ and $[AB]$ respectively.

(a) Explain why F, B, C and E (in that order) are the vertices of a Saccheri quadrilateral and give reasons for your answer.

(b) Are the corresponding statements for A, F, E and D (in that order) true or false for each of Euclidean and hyperbolic geometry? Give reasons for your answers in **both** cases.

SOLUTION

(a) The lines $FB = AB$ and $CE = CD$ are both perpendicular to BC, and $|FB| = \frac{1}{2}|AB| = \frac{1}{2}|CD| = |CE|$, so the conditions for a Saccheri quadrlilateral are satisfied.



(b) **Euclidean case.** We claim that A, F, E and C (in that order) **ARE** the vertices of a Saccheri quadrilateral. The midpoint condition implies that $|FA| = \frac{1}{2}|AB| = \frac{1}{2}|CD| = |DE|$, and the Saccheri quadrilateral in (a) is a rectangle because we have assumed the plane is Euclidean. In particular, this implies that the line EF is perpendicular to each of $AF = AB$ and $CD = DE$. The conclusions of the preceding two sentences imply that A, F, E and C (in that order) are the vertices of a Saccheri quadrilateral.■

Hyperbolic case. We claim that A, F, E and C (in that order) **ARE NOT** the vertices of a Saccheri quadrilateral. In fact, EF is not perpendicular to either $AF = AB = BF$ or $CE = CD = DE$ because (a) implies that $|\angle BFE| = |\angle CEF| < 90°$ in hyperbolic geometry.■

6. [*20 points*] Assume we are working in hyperbolic plane geometry and we are given a right triangle $\triangle ABC$ with a right angle at C, and let $D \in (AB)$ be the foot of the perpendicular from C to AB. Prove that $|\angle BAC| < |\angle BCD|$.

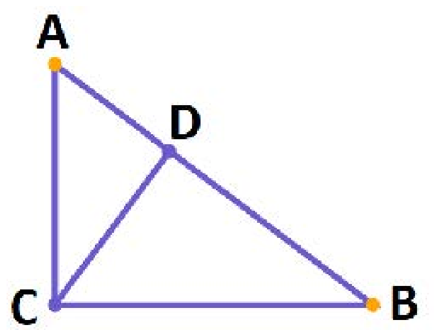
SOLUTION

If $\triangle XYZ$ has a right angle at Y, then

$$\delta(\triangle XYZ) \;=\; 180 \;-\; |\angle YXZ| \;-\; 90 \;-\; |\angle YZX| \;-$$

$$90 \;-\; |\angle YXZ| \;-\; |\angle YZX|$$

and by the additivity of angle defects we know that $\delta(\triangle CDB) < \delta(\triangle ACB)$.



Since $\angle ABC = \angle CBD$, if we let θ denote the measure of this angle we may rewrite the angle defect inequality as

$$90 \;-\; \theta \;-\; |\angle DCB| \;=\; \delta(\triangle CDB) \;<\; \delta(\triangle ACB) \;=\; 90 \;-\; |\angle CAB| \;-\; \theta$$

which yields the desired inequality $|\angle BAC| < |\angle BCD|$.■