Undefined concepts

Euclid tried to define everything — not formally possible

2D geometry — Assume given a set P, call its members points. (Nonempty!)

nonempty

Also assume we are given a family of

proper

subsets we shall call lines. $\mathcal{L} =$ proper subsets

Assume the following axioms (Incidence)

(I1) Given two distinct points, there is a unique line containing them.

(I2) Every line contains ≥ 2 points.

Models for axioms   Ideally, ordinary geometry

Also, S = set with $\geq$ **3** elements, $\mathcal{L}$ = all subsets with exactly two elements.

... and many more ...

NOTATION xy = line cont. $x + y$

Only one noteworthy theorem.

If $L \neq M$ are lines in P, then $L \cap M$ has at most one point.

Note: We can also prove that P has at least 3 points as follows: There is at least one line in P, and it has at least two points. But there is also a point in P not on L. This yields a third point of P.

The proof shows the contrapositive of the thm., and the latter is equiv. to the stated theorem.

Proof Say $x \neq y$, both in $L \cap M$.

By I3 there is only one line N so $x, y \in N$. But L & M have this property. So must have $L = M$.

COLLINEAR SET A, $A \subseteq L$, some line.

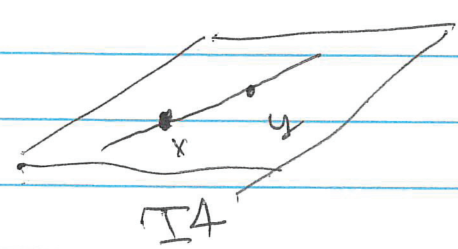
3D geometry $\mathcal{S}$ nonempty set
$\mathcal{L}$ = proper subset family
$\mathcal{P}$ = another one, disjoint from $\mathcal{L}$

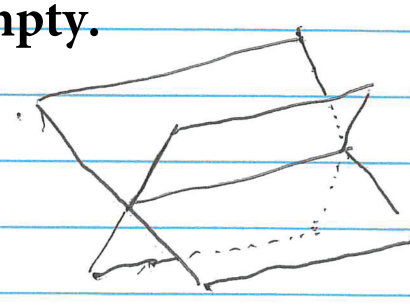
First two axioms plus

(I3) Given a noncollinear subset $\{x, y, z\} = B$, there is a unique plane P containing B.

(I4) If x, y distinct in P, then (line) $xy \subseteq P$.

(I5) If P and Q are planes such that $P \neq Q$ then $P \cap Q$ is a line. or empty.



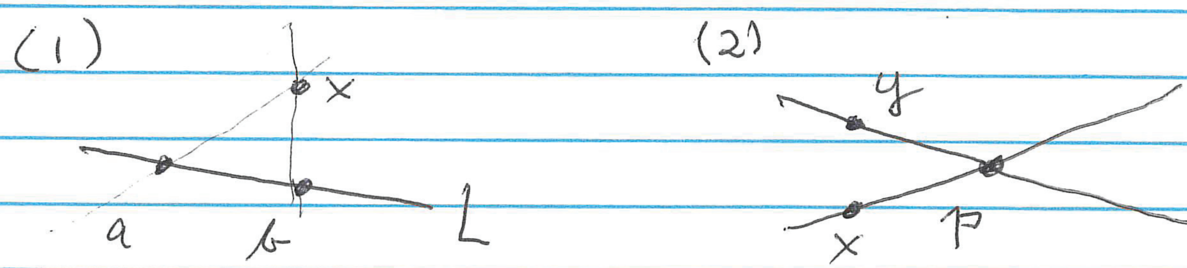


(I6) Every plane has ≥ 3 points.

As in the 2-dim case, in the 3-dim case we can show that S has at least 4 points. There is a plane P in S and it has three points, and there is a fourth point which is in S but not in P.

Two consequences (1) Given line L and $x \notin L$, there is a unique plane P so $L \subseteq P$ and $x \in P$.

(2) If two distinct lines meet at a single pt., then they determine a plane.



Pictures make logic transparent, but are not proofs themselves.!

Now we need to add more data

Distance Postulates

Ruler Postulate

Distance $d: S \times S \to [0, \infty)$ given

$d(x,y) \geq 0$, equality $\Leftrightarrow$ $x = y$

$d(x,y) = d(y,x)$. [No triangle inequality yet!]

Ruler: There are 1-1 correspondences

$f: L$ (each line) $\longleftrightarrow \mathbb{R}$ so that

$d(x,y) = |f(x) - f(y)|$.

(Placement)

Strong Ruler Property

Given $x \neq y$ in L, can find f so that $f(x) = 0$, $f(y) > 0$.

How to derive this conclusion Take any f_0

Let $\varepsilon_0 = 1$ if $f_0(x) < f_0(y)$, $\varepsilon_1 = -1$ otherwise.

Check that $g(t) = \varepsilon_0 [f_0(t) - f_0(\mathbf{x})]$ is 1-1

and $d(t_1, t_2) = |g(t_1) - g(t_2)|$.

1-1 $g(t_1) = g(t_2) \Rightarrow \varepsilon_0 [f_0(t_1) - f_0(x)] = \varepsilon_0 [f_0(t_2) - f_0(x)]$

$\Rightarrow f_0(t_1) - f_0(x) = f_0(t_2) - f_0(x) \Rightarrow f_0(t_1) = f_0(t_2) \Rightarrow$

$t_1 = t_2$ since f_0 is 1-1.

onto Let $a \in \mathbb{R}$. Want $a = [f_0(t) - f_0(x)]\varepsilon_0$ some t

Well, $\varepsilon_0 a = f_0(t) - f_0(x)$, $\varepsilon_0 a + f_0(x) = f(t)$

Suggests the right choice of t. Substitute (work backwards) to verify that $a = g(t)$.

distance preserving $d(t_1, t_2) = |f_0(t_1) - f_0(t_2)|$ is given.

$$\text{R.H.S.} = \left| [f_0(t_1) - f_0(x)] - [f_0(t_2) - f_0(x)] \right|$$

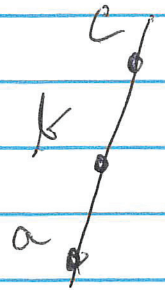
and $|\varepsilon_0| = 1$ means this is

$$\left| \varepsilon_0 \left([f_0(t_1) - f_0(x)] - [f_0(t_2) - f_0(x)] \right) \right| =$$

$$\left| \varepsilon_0 [f_0(t_1) - f_0(x)] - \varepsilon_0 [f_0(t_2) - f_0(x)] \right| =$$

$$|g(t_1) - g(t_2)|.$$

Why do we need this?



Given three points on a line, we "see" that one is between the other two. This was only discussed casually in Euclid, but it is absolutely essential for a logically sound treatment of classical geometry.

<u>RECALL</u> When is $|a+b| = |a| + |b|$ in $\mathbb{R}$?

Either $a, b \geq 0$ <u>or</u> $a, b \leq 0$.

<u>Define</u> $a * b * c$ (b is <u>between</u> a & c) $\Longleftrightarrow$

$d(a,c) = d(a,b) + d(b,c)$.

This gives us all we need. The following properties are "obvious" but must be verified.

Betweenness and ruler functions

Theorem On line L with ruler function f,

$a * b * c \iff f(a) < f(b) < f(c)$ or $f(a) > f(b) > f(c)$.

Derivation $|p+q| = |p| + |q| \iff$ $p, q \geq 0$ or $p, q \leq 0$. Now let $p = f(a) - f(b)$ & $q = f(b) - f(c)$, so $p+q = f(a) - f(c)$ and $|p+q| = |p| + |q| \iff d(a,c) = d(a,b) + d(a,c)$. The case $p, q \geq 0$ corresponds to $f(a) > f(b) > f(c)$, and $p, q \leq 0$ corresponds to $f(a) < f(b) < f(c)$. ∎

Theorem Given distinct p, q & r on L, one & only one is between the other two.

Proof Six cases

$f(p) < f(q) < f(r)$	$f(q) < f(r) < f(p)$	$f(q) < f(p) < f(r)$
$f(p) > f(q) > f(r)$	$f(q) > f(r) > f(p)$	$f(q) > f(p) > f(r)$
q between	r between	p between

The following are typical applications which are needed for more substantial results.

(1) $a*b*c$ & $b*x*c \Rightarrow a*x*c$

First Find ruler fcn so $f(a)<f(b)$.

By previous, have $f(a)<f(b)<f(c)$.

Now $b*x*c \Rightarrow$ either $f(b)<f(x)<f(c)$ or $f(b)>f(x)>f(c)$. Second violates previous conclusion, so $f(a)<f(b)<f(x)<f(c)$.

(2) $a*x*c$ + $a*y*c \Rightarrow a*x*y$ or $a*y*x$.

Choose ruler $f(a)=0$, $f(c)>0$.

$0<f(x)<f(c)$

$0<f(y)<f(c)$ Why?

So either $0<f(x)<f(y)$ or $0<f(y)<f(x)$.

See lecture02a.pdf for a cleaned up version of this material.

Dictionary

$A \neq B$, f ruler with ~~$f(a)=0$~~, $f(b)>f(a)$

closed segt $[AB]$	$x=a,b$ or $a*x*b$	$f(x)\in[f(a),f(b)]$
open segt (AB)	$a*x*b$	$f(x)\in(f(a),f(b))$
closed ray $[AB$	$x=a$, $a*x*b$, $x=b$, $a*b*x$	$f(x)\in[f(a),\cancel{f(b)}\ \infty)$
open ray $(AB$	$x\in[AB-\{A\}$	$f(x)\in[-\infty,f(a)]$
opp closed ray $[AB^{op}$	$x=A$ or $x\in L-[AB$	$f(x)\in(-\infty,f(a)]$
opp ~~closed~~ open ray $(AB^{op}$	~~$x\in[AB^{op}-\{A\}$~~ $x\in L-[AB$	$f(x)\in(-\infty,f(a))$.