Appendix B to Lecture 02:

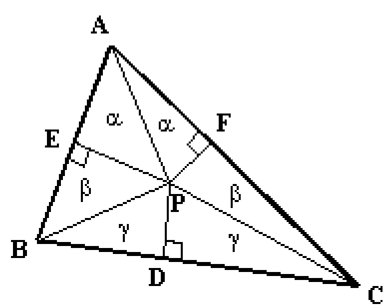
The isosceles triangle fallacy

The treatment below is adapted from the following source:

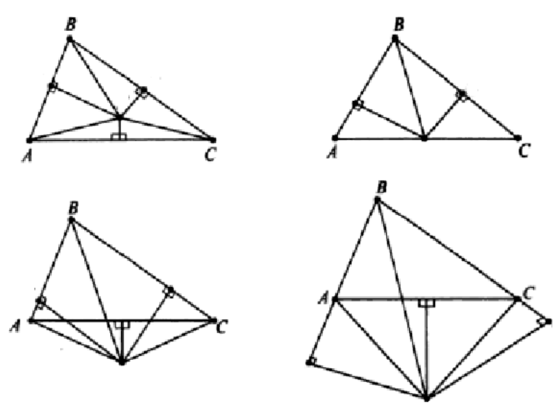
http://www.mathpages.com/home/kmath392.htm

One well – known illustration of the logical fallacies to which classical methods of Euclid and others are vulnerable is a "proof" that all triangles are isosceles. *The discussion below explicitly assumes that the reader is familiar with some basic ideas from elementary (high school) geometry*.

Given an arbitrary triangle $\triangle$**ABC,** draw the angle bisector of the interior angle at **A,** and draw the perpendicular bisector of the closed segment **[BC]** with midpoint **D,** as shown below:



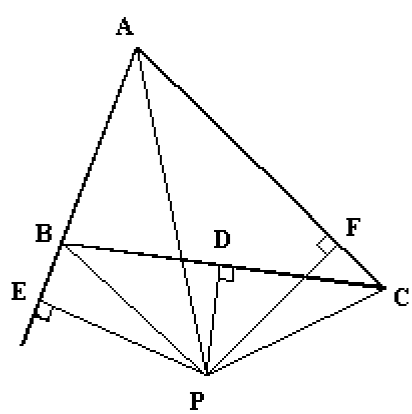
If the angle bisector at **A** and the perpendicular bisector of **[BC]** are parallel or identical, then $\triangle$**ABC** is isosceles (this is a valid result in Euclidean geometry and can be shown directly by standard methods; we shall omit the details because they do not involve the fallacious argument). On the other hand, if the lines in questions are not parallel, then they intersect at a point, which we shall call **P.** In the drawing, this point lies in the interior of the triangle, so we need to consider cases where **P** lies on or outside the triangle. Drawings for various cases are given below:



We shall now focus on the case where **P** lies indside the triangle; the same argument applies to the cases in the upper right and lower left cases. We can drop the perpendiculars from **P** to **AB** and **AC,** which will meet these lines at **E** and **F** respectively**.** Now the two triangles △**APE** and △**APF** have equal angles and share a common side**,** so they are congruent**.** Therefore **|PE|** = **|PF|.** Also**,** since **D** is the midpoint of **[BC],** the triangles △**PDB** and △**PDC** are congruent right triangles**,** and hence **|PB|** = **|PC|.** From this it follows that the triangles △**PEB** and △**PFC** are congruent to each other**,** so that we must have **|BA|** = **|BE|** + **|EA|** = **|CF|** + **|FA|** = **|CA|.** In the case at the lower right, we have **|BA|** = **|BE|** – **|EA|** = **|CF|** – **|GA|** = **|CA|.** Therefore in all these cases △**ABC** must be isosceles.**(■ Really???)**

<u>This conclusion is obviously absurd, but where is the mistake?</u>

The key to understanding the problem is to scrutinize the drawing upon which the incorrect reasoning was based**,** and if we construct the points and lines described in this proof more carefully and accurately**,** we see that the actual configuration doesn't look like the picture above**.** It turns out that the point **P** must lie outside the triangle △**ABC.** However**,** if we carry out the proof on this basis**,** and if we now assume the points **E** and **F** also fall outside the triangle**,** we still conclude that the triangle is isosceles**.** This too is an incorrect configuration**.** The ***actual configuration*** of points given by the stated construction is for the point **P** to be outside the triangle △**ABC,** and for exactly one of the points **E, F** to be between the vertices of the triangle**,** as shown below**:**



We still have **|AE|** = **|AF|, |PE|** = **|PF|,** and **|PB|** = **|PC|,** and it still follows that **|BE|** = **|FC|,** but now we see that even though **|AE|** = **|AF|** and **|BE|** = **|FC|** it does ***<u>not</u>*** follow that **|A, B|** = **|AC|,** for even though **F** is between **A** and **C,** the point **E** is ***<u>not</u>*** between **A** and **B.**

The moral of this discussion is that **<u>we must be careful and systematic when using betweenness in a geometrical proof</u>.**

Actually**,** the argument above does yield valid **(**but different and far less alarming**)** conclusions**.** For example**,** if sides **[AB]** and **[AC]** have unequal lengths**,** then either **E** does not lie on **[AB]** or else **F** does not lie on **[AC].** For if both were true the argument above would show that △**ABC** is isosceles**,** and we have assumed this is false**.**