Math 133
Fall 2021

MORE SOLUTIONS FOR WEEK 01 EXERCISES

Assume that $(\mathbf{S}; \mathcal{P}; \mathcal{L}; d; \alpha)$ or $(\mathbf{P}; \mathcal{L}; d; \alpha)$ is a system which satisfies the Space or Plane Incidence Axioms and also the Ruler Axiom.

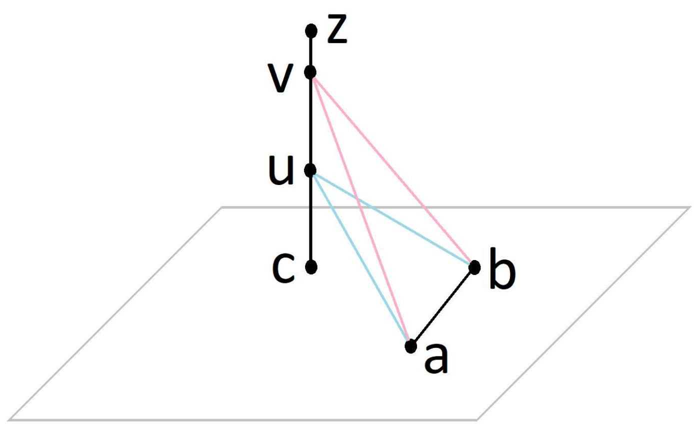
5. A point X lies on AB if and only if $X = A$, $A = B$, $A * X * B$, $X * A * B$, or $A * B * X$. In all but the last case, $X \in [BA$, and in the last case $X \in [AB$. Therefore $X \in [AB \cup [BA$ and hence $AB \subset [AB \cup [BA$.

To verify the reverse inclusion, note that $[AB$ and $[BA$ are subsets of AB and hence $[AB \cup [BA \subset AB$.■

6. Choose a ruler function $f : AB \to \mathbb{R}$ such that $f(A) < f(B)$. Then $A * B * C$ implies that $f(A) < f(B) < f(C)$. Now $X \in [BC$ implies that $f(X) \geq f(B) > f(A)$ and hence $X \in [AB$. Furthermore, $A * X * B$ implies $f(A) < f(X) < f(B)$, so that $X \in [AB$ in this case too. Therefore $[AB \supset [AB] \cup [BC$.

To verify the reverse inclusion, note that $X \in [AB$ if and only if $f(X) \geq f(A)$. If $f(X) \leq f(B)$ then $X \in [AB]$, while if $f(X) > f(B)$ then $X \in [BC$. In either case if follows that $X \in [AB] \cup [BC$ and hence $[AB \subset [AB] \cup [BC$.■

7. Here is a sketch depicting the hypotheses.



Suppose that the planes **abu** and **abv** are equal, and let Q denote this plane; by construction we have $\mathbf{ab} \subset Q$. Since $\mathbf{u}$ and $\mathbf{v}$ are distinct points on the line $\mathbf{cz}$, it follows that $\mathbf{cz} \subset Q$. And since there is a unique plane containing $\mathbf{a}, \mathbf{b}, \mathbf{c}$ it follows that $P = Q$. But the latter implies $\mathbf{z} \in P$, contradicting our hypotheses. The source of the contradiction is the assumption that the two planes are equal, and hence they must be distinct.

To conclude, we must also show that neither of these planes is equal to P; since the roles of $\mathbf{u}$ and $\mathbf{v}$ are interchangeable in the problem, it is enough to show $P \neq \mathbf{abu}$. Assume the planes are equal. Then $\mathbf{c}$ and $\mathbf{u}$ both lie in P, and hence the line $\mathbf{cu} = \mathbf{cz}$ is contained in P. This contradicts our assumption that $\mathbf{z}$ is not in P. The source of the contradiction is the assumption that the two planes are equal, and hence they must be distinct.■

Note. Since the Ruler Postulate implies that every line has infinitely many points, it follows that there are infinitely many planes containing **ab** when this postulate holds.

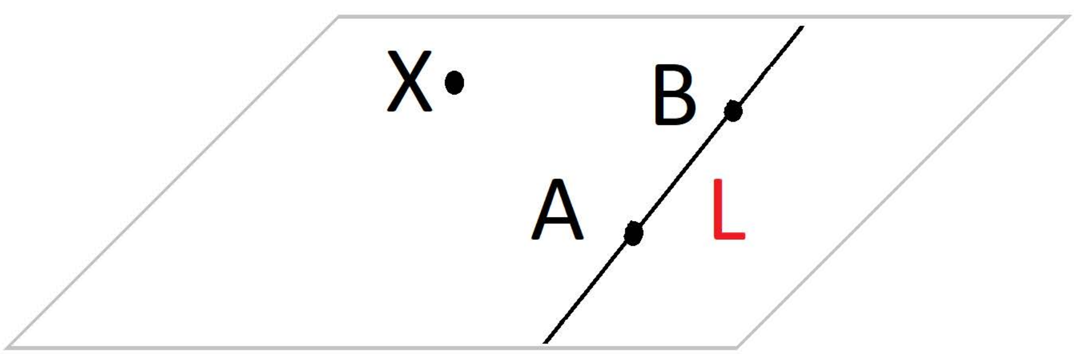
8. If P_1, P_2 and P_3 are distinct planes in a 3-space **S**, show that their intersection is either the empty set, a single point or a line. Describe examples in ordinary geometry for which the intersections are of each (mutually exclusive) type.

We have to analyze cases where the intersection contains at least two points. Suppose that $A \neq B$ and both lie in $P_1 \cap P_2 \cap P_3$. Then the line AB also lies in this intersection by one of the Incidence Axioms. If a point C not on AB also lies in the intersection, then there is a unique plane containing AB in C. Since P_1, P_2 and P_3 all contain this line and external point, it follows that $P_1 = P_2 = P_3$. Therefore, if we have three distinct planes and they have more than one point in common, they they have a line in common.

Here are examples in coordinate 3-space: (0) The planes defined by $y = i$ (where $i = 1, 2, 3$) have no points in common. (1) The intersection of the planes defined by $x = 0$, $y = 0$ and $z = 0$ is just the origin (0,0,0). (2) The intersection of the planes $x = 0$, $y = 0$ and $x = y$ is the z-axis.■

9. Suppose to the contrary that the lines meet at some point X. Since there is a unique plane containing two intersecting lines, it follows that the AB and CD both lie on this plane, contradicting the assumption that A, B, C, D are noncoplanar. Therefore assumption in the first sentence is false, and this means the two lines must be disjoint.■

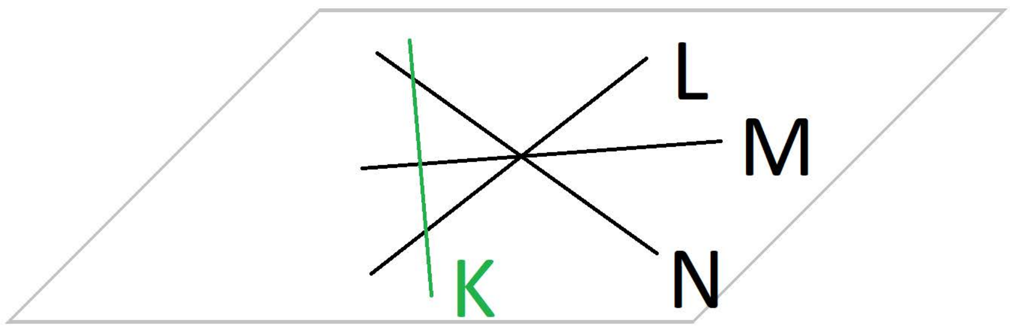
10. First of all, here is a sketch.



Follow the hint, and let A and B be two points of L. Then A, B and X are noncollinear, for if M were a line containing all three then M would have to be equal to L by uniqueness of a line through two points. Let P be the unique plane containing A, B and X. The the Incidence Axioms imply that $L = AB \subset P$, so P is a plane containing L and X. This proves existence.

To prove uniqueness, if Q is a plane such that $X \in P$ and $L \subset Q$, then we automatically have $A, B, X \in Q$. Since there is a unique plane containing these three noncollinear points it follows that $Q = P$.■

11. Here is a sketch reflecting the hypotheses.



We know that there is a plane P containing L and M. If $A \in K \cap L$ and $B \in K \cap M$, then $A, B \in P$ implies that $K = AB \subset P$. Therefore if $C \in K \cap N$ we have $C \in N$. By construction $N = CX$, and since $C, X \in P$ we have $N \subset P$. Therefore P contains all four lines.■

12. The hypotheses imply that $d(A,B) + d(B,C) = d(A,C) = d(X;Z) = d(X,Y) + d(Y,Z)$ and $d(B,C) = d(Y,Z)$. Therefore $d(A,B) = d(A,C) - d(B,C) = d(X,Z) - d(Y,Z) = d(X,Y)$.■