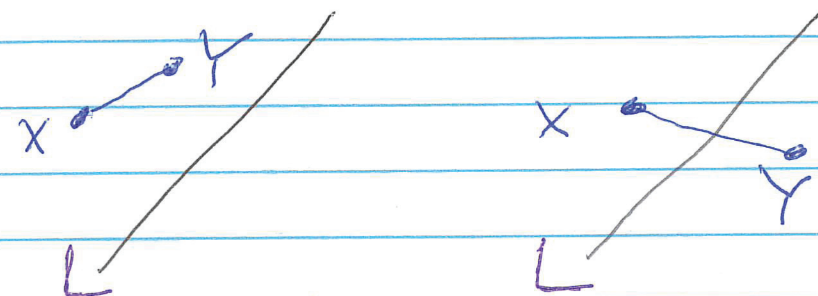
Plane separation

A concept related to betweenness, also discussed casually in Euclid but requires a logically sound treatment.

Plane P, line L ⊆ P, X & Y in P but not L



same side of L   opposite sides of L

Notice the second holds ⟺ the lines L and XY meet at a point between X & Y.

<u>Plane Separation Postulate</u> L & P as above. Then P − L is a union of two nonempty disjoint sets A & B such that

(1) If X & Y in either A or B, then (XY) ⊆ the appropriate set (A & B are convex)

No dents or holes in a convex set

neither convex

(2) $X \in A$ & $Y \in B \Rightarrow L \cap (XY) \neq \phi$.

<u>Space separation postulate</u> Similar, but replace $\begin{Bmatrix} \text{line} \\ \text{plane} \end{Bmatrix}$ with $\begin{Bmatrix} \text{plane} \\ \text{space} \end{Bmatrix}$.

The second implies the first for all planes in the 3-space (not to be used very much).

Algebra yields line separation: L line, $X \in L$. Then $L - \{X\} =$ two disjoint open rays $(AB$ & $(AC$

s.t. (1) X, Y on one open ray $\Rightarrow (XY) \subseteq$ open ray
(2) X in one, Y in other $\Rightarrow X * A * Y$.
(Use ruler function!)

<u>Theorem</u> L & M lines meeting at X, Y & Z in $M - \{X\}$.

(1) Y, Z on same side of $L \Leftrightarrow$ either $X * Y * Z$ or $X * Z * Y$.

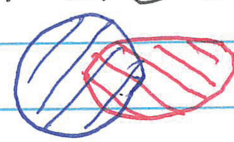
(2) Y, Z on opposite sides $\Leftrightarrow Y * Z * X$.

<u>Verification</u> One of X, Y, Z is between the other two (and only one). Enough to check (2) is true.

($\Rightarrow$) True by Plane Separation Postulate.
($\Leftarrow$) Suppose on same side. Which of

$X*Z*Y$, $X*Y*Z$, $Z*X*Y$ is true?
Why not the last one? Z, Y on same side
$\Rightarrow (ZY) \subseteq$ that side $\Rightarrow X$ is too. But
the side is contained in $L - \{X\}$, contradiction.

__Theorem.__ If A & B are convex, so is $A \cap B$.



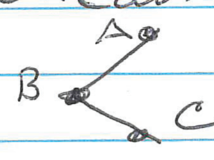
__Proof__ Let $X, Y \in A \cap B$.

$$\left.\begin{array}{l} \text{Then } X, Y \in A \Rightarrow (XY) \subseteq A \\ \text{———"——— } B \Rightarrow (XY) \subseteq B \end{array}\right\} \text{Hence } (XY) \subseteq A \cap B.$$

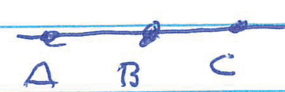
Suffices to define other regions (convex).

__ANGLES.__ A, B, C noncollinear.

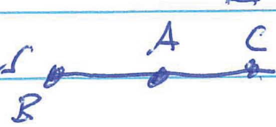
$\measuredangle ABC = \overrightarrow{BA} \cup \overrightarrow{BC}$



For us, straight angles



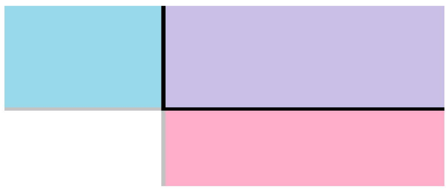
and degenerate angles



are __NOT__ angles!!

Interior of an angle

Int $\measuredangle ABC$ = A-side BC $\cap$ C-side AB



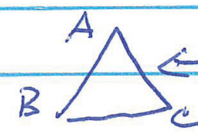
← interior (edges not included!)

Simple consequence: If $X \in (AC)$ then $X \in$ Int $\measuredangle BAC$.

Proof: $A * X * C \Rightarrow$ X & A on same side BC
X & C on same side AB.

Exercise Same conclusion if $Y \in (BX$.
(Y & X same side of BC and AB)

Interior of triangle ΔABC =
$[AB] \cup [BC] \cup [AC]$

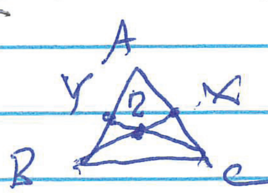


← Hollow!

Int ΔABC =

Int $\measuredangle ABC \cap$ Int $\measuredangle BCA \cap$ Int $\measuredangle CAB$

Exercise It's the intersection of any two.

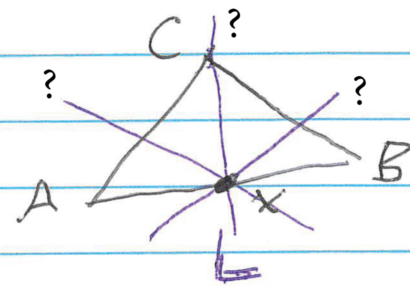


$A * X * C$
$A * Y * B$
$Z \in (CX) \cap (BY)$

To prove: $Z \in$ Int ΔABC.

Two more substantial results IN THE PLANE

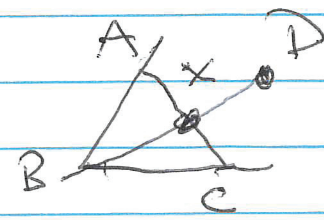
Pasch's Theorem: Given $\triangle ABC$ and L such that $L \cap (AB) = \{X\}$. Then one of the following is true: (1) $L \cap (BC) \neq \phi$, (2) $C \in L$ (3) $L \cap (AC) \neq \phi$



Used in Euclid w/o assuming it.

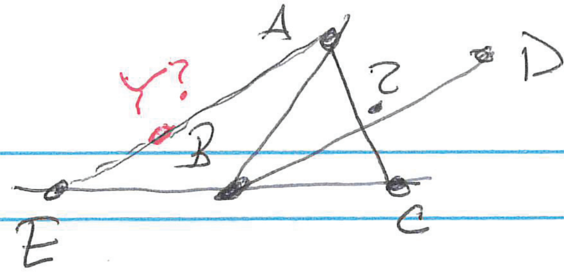
Proof. Three options for C: (on opp side L as B) (same side of L as B) $C \in L$. Analyze first two cases. Opp sides: $(BC) \cap L \neq \phi$ by Plane Separation. Same side. B & A on opp sides since $B * X * C$. Hence C & A on opp sides also. But then $L \cap (AC) = \phi$.

Crossbar Theorem.



Given $D \in \text{Int} \measuredangle ABC$. Then there is a point $X \in (BD \cap (AC)$.

Proof.



$E * B * C$

Then BD meets either (AC) or (AE).
Note $A \notin BD$ since A, B, D not collinear by hyp.
Need to show $BD \cap (AE) = \emptyset$. Say $Y \in BD \cap (AE)$. Then E, Y on same side of AB.
So Y on opp side AB as $D + C$ ($D \in$ Interior!)

INSERT: and since Y, B, D are collinear, the common point must be B

Hence (YD) meets AB, so D & Y on opp sides of BC. However $A*Y*E \Rightarrow Y, A, D$ all on same side of BC! Contradiction. Source? Assuming $BD \cap AE \neq \emptyset$. Hence $BD \cap (AC) \neq \emptyset$. Let X be this pt.

Finally, must show $X \in (BD$.

$A*X*C \Rightarrow A$ & X on same side BC
C & X on same side AB

But also $A + D$ on same side BC, so likewise for D and X. By previous warmups, $X \in (BD$.

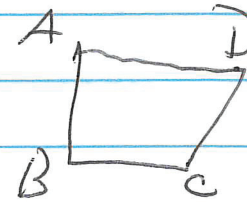
First, try to understand passively, then actively as much as possible!

A, B, C, D no 3 collinear

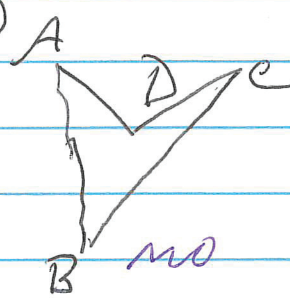
Diagonal Theorem.

Say A, B, C, D form the vertices of a convex quadrilateral if:

A, B on same side CD
B, C —— " —— AD
C, D —— " —— AB
D, A —— " —— BC



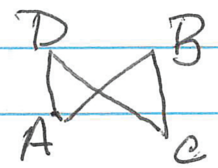
yes



no

Note $\square ABCD$ is not a convex set!

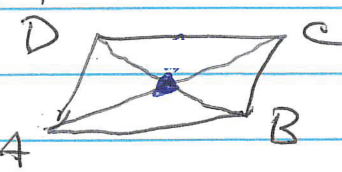
Two edges disjt. or have ONE common endpt.



also no.

$\square ABCD = [AB] \cup [BC] \cup [CD] \cup [AD]$

Thm. Given $\square ABCD$, there is a point in $(AC) \cap (BD)$.



Proof. Use Crossbar Thm. twice.

Hypotheses $\Longrightarrow C \in \text{Int} \measuredangle DAB = \text{D-side } AB \cap \text{B-side } AD$

So $(AC$ meets (BD) in some point X.

Also $(BD$ meets (AC) in some point Y.

Now $AC \neq BD$ by def of convex quad.

Hence $AC \cap BD$ has at most one point.

Since $X, Y \in AC \cap BD$, must have $X = Y$ and $X \in (AC)$.