Angle measurement

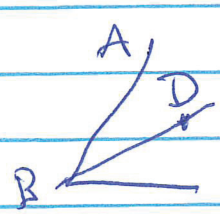
Axioms Given A, B, C noncollinear, there is a number $|\measuredangle ABC|$ between 0 and 180 with the following properties:

(AM0) If $D \in (BA$ & $E \in (BC$, so that $\measuredangle ABC = \measuredangle DBE$ $\boxed{\text{as sets}}$ then $|\measuredangle ABC| = |\measuredangle DBE|$.

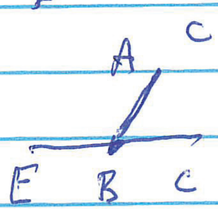
Important to distinguish between $|\measuredangle ABC| = |\measuredangle XYZ|$ and (stronger) $\measuredangle ABC = \measuredangle XYZ$!

Also $|\measuredangle ABC| = |\measuredangle CBA|$.

(AM1) PROTRACTOR POSTULATE. If $0 < x < 180$ and A, B, D noncollinear, then there is a unique ray $(BC \subseteq D$-side AB so $|\measuredangle ABC| = x$.



(AM2) ADDITIVITY $D \in \text{Int} \measuredangle ABC \Longrightarrow$ $|\measuredangle ABC| = |\measuredangle ABD| + |\measuredangle DBC|$.



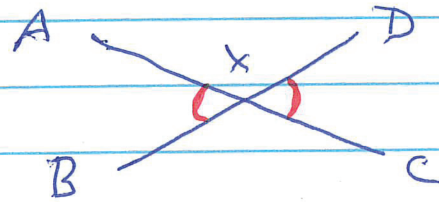
(AM3) SUPPLEMENT PROPERTY $E * B * C$ & $A \notin BC \Longrightarrow |\measuredangle ABC| + |\measuredangle ABE| = 180$.

Recall $[BC \cup [BE$ not an angle in our sense. Likewise for $[BD \cup [BC$ if $[BD = [BC$.

Consequences

Vertical Angle Theorem. Given $A * X * C$ and $B * X * D$. Then $|\measuredangle AXB| = |\measuredangle CXD|$.

$AC \neq BD$



Proof (One of the earliest)

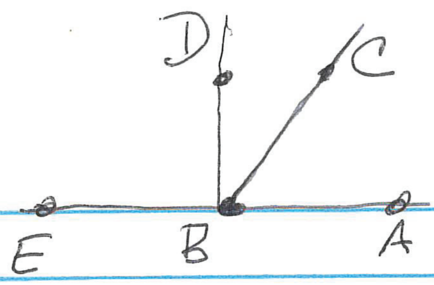
$$|\measuredangle AXB| + |\measuredangle AXD| = 180 = |\measuredangle CXD| + |\measuredangle AXD|.$$

by Supp Property. Subtract $|\measuredangle AXD|$ from the left and right sides.

Angle measure comparison: C, D on same side of AB. Then $|\measuredangle DAB| < |\measuredangle CAB| \Leftrightarrow$ $D \in \operatorname{Int} \measuredangle CAB$.

Proof: Protractor $\Rightarrow$ (equal measures $\Leftrightarrow$ $[AC = [AD$)
Additivity $\Rightarrow$ ($|\measuredangle DAB| < |\measuredangle CAB|$ if $D \in \operatorname{Int} \measuredangle CAB$).

Converse



$\boxed{\text{Given}}$

$D \notin [BC \cup$ A-side BC

$E*B*A$

A & E on opp sides BC since $E*B*A$
Hence D & A opp sides $\Rightarrow$ D & E same side.
Hence $D \in \text{Int}\,\triangle EBC$ by def of interior.

This means $|\measuredangle DBE| < |\measuredangle CBE|$ by additivity.
Apply Supp Property to conclude

$$180 - |\measuredangle ABD| < 180 - |\measuredangle ABC|$$

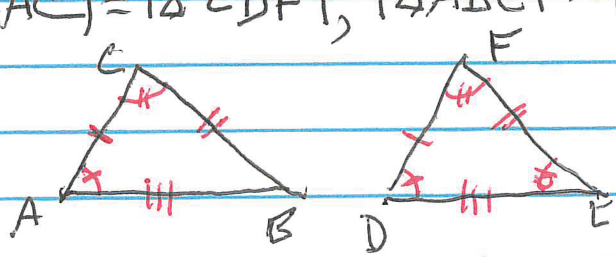
By algebra $|\measuredangle ABC| < |\measuredangle ABD|$.

Exercise Show $C \in \text{Int}\,\measuredangle ABD$ using these ideas.

Relating linear and angular measure.

Congruence axioms Write $\triangle ABC \cong \triangle DEF$

if $|AB| = |DE|$, $|AC| = |DF|$, $|BC| = |EF|$ and
$|\measuredangle BAC| = |\measuredangle EDF|$, $|\measuredangle ABC| = |\measuredangle DEF|$, $|\measuredangle ACB| = |\measuredangle DFE|$



The vertizes must be ordered consistantly in the statement!!

In other words $\triangle ABC \cong \triangle DEF$ is <u>not</u> equivalent to $\triangle ABC \cong \triangle DFE$, etc., but it is equivalent to $\triangle ACB \cong \triangle DFE$.

<u>THREE AXIOMS.</u> If any of

(SAS) $|\measuredangle ABC| = |\measuredangle DEF|$, $|AB| = |DE|$ $|AC| = |DF|$,

(ASA) $|\measuredangle ABC| = |\measuredangle DEF|$, $|AB| = |DE|$, $|\measuredangle BAC| = |\measuredangle EDF|$,

(SSS) $|AB| = |DE|$, $|AC| = |DF|$, $|BC| = |EF|$

hold, then $\triangle ABC \cong \triangle DEF$.

In fact, any one of these implies the other two, but no attempt to do so in this course.

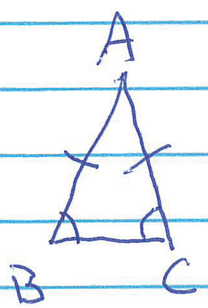
Without something like this, linear + angular measurement might have nothing to do with each other.

Euclid argued the first is true by asserting one triangle can be moved to be nicely placed with respect to the other. But he never made any assumptions about a concept of rigid motion!

<u>Properties of $\cong$</u> $\triangle ABC \cong \triangle ABC$, $\triangle ABC \cong \triangle DEF \Rightarrow \triangle DEF \cong \triangle ABC$, $\triangle ABC \cong \triangle DEF$ and $\triangle DEF \cong \triangle GHK \Rightarrow \triangle ABC \cong \triangle GHK$ (equivalence relation).

We allow $\{A, B, C\} = \{D, E, F\}$.

<u>Isosceles Triangle Theorem</u> In $\triangle ABC$, $|AB| = |AC| \Leftrightarrow |\measuredangle ABC| = |\measuredangle ACB|$.



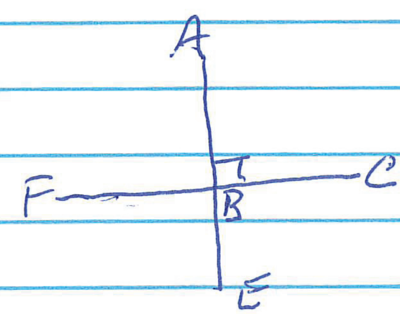
<u>Proof</u> $(\Rightarrow)$ $\triangle ABC \cong \triangle ACB$ because $\measuredangle ABC = \measuredangle ACB$ and SAS. $(\Leftarrow)$ Same conclusion since $[BC] = [CB]$ and ASA.

<u>PERPENDICULARITY</u> Lines $AB \neq BC$

$AB \perp BC$ means $|\measuredangle ABC| = 90°$.

<u>Key Property</u> Suppose $AB \perp BC$ and $A * B * E$, $F * B * C$.

Then $|\measuredangle ABF| = |\measuredangle EBC| = |\measuredangle FBE| = 90°$



<u>Proof</u> $|\measuredangle FBA| + |\measuredangle ABC| = 180° \Rightarrow |\measuredangle FBA| = 90°$

$|\measuredangle EBC| + |\measuredangle ABC| = 180° \Rightarrow |\measuredangle EBC| = 90°$

$90° = |\measuredangle ABC| = |\measuredangle EBF|$ by Vertical Angle Thm.

Fundamental Existence Property L line in plane P, $B \in P \Rightarrow$ there is a unique line M so $B \in M$ and $L \perp M$.

Two cases $B \in L$ and $B \notin L$

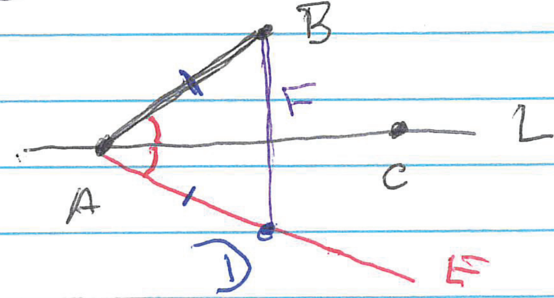
Proof when $B \in L$ Choose one side of $L = BC$

Then there is a unique ray $[BA$ so $|\measuredangle ABC| = 90°$ proving existence. Must also show uniqueness. Let $D \in P - L$ so $|\measuredangle DBC| = 90°$ If D & A on same side, then we also have $[BD = [BA$, so $BD = BA$ & $\measuredangle DBC = \measuredangle ABC$. On the other hand, if D & A on opposite sides, let E satisfy $D*B*E$.

Then E & A on same side, so $[BE = [BA$ & $BE = BA = BD$.

Proof when $B \notin L$: Let $L = AC$.

It's complicated



Consider ray $[AE$ so $E \in$ op B-side AC and $|\measuredangle EAC| = |\measuredangle BAC|$. Take $D \in (AE$ so $|AB| = |AD|$.

If $AB \perp AC$ have a perpendicular.

Otherwise, proceed as follows:

(BD) meets AC in some $F (\neq A)$. There is a common pt. by plane separation. If $F = A$, then $AB = AD = BD$, so $B * A * D$ (opp sides) and $|\measuredangle ABC| = 90°$. So assume $F \neq A$ henceforth.

Not covering all cases is a common mistake!

Now either $F \in (AC$ or else $F * A * C$.

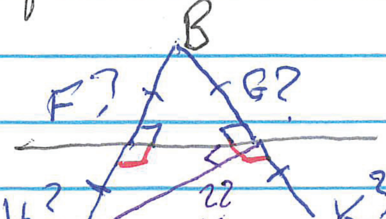
In the first case $[AF = [AC$ so $\measuredangle BAF = \measuredangle BAC$ and $\measuredangle DAF = \measuredangle DAC$. In the second, $|\measuredangle BAF| = 180 - |\measuredangle BAC| = 180 - |\measuredangle DAC| = |\measuredangle DAF|$.

In either case $|\measuredangle BAF| = |\measuredangle DAF|$.

By SAS, $\Delta BAF = \Delta DAF$, hence $|\measuredangle BFA| = |\measuredangle DFA|$. Since $B * F * D$, it follows that $180 = |\measuredangle BFA| + |\measuredangle DFA| = 2|\measuredangle BFA|$, so $|\measuredangle BFA| = 90$.

Hence $BF \perp FC = AC$. This proves existence.

Uniqueness? $BF, BG \perp L$, so $L = FG$ if $F \neq G$.



Blue = hypothetical argt. which yields contradiction

Then $|BF| = |BG|$ (Isosceles)

Take H, K so that $B * F * H$, $B * G * K$ and $|FH| = |BH| = |BG| = |GK|$.

$\Delta BFG \cong \Delta HFG$ by SAS. Hence $HG \perp BG$. But $FG \perp BG$, so $HG = FG$. Impossible since $H \in BF$, $H \neq F$ and $BF \cap FG = \{F\}$.