Inequalities for triangles

classical "locus" = modern "set"

"Locus Theorem" $A \neq B$ in plane P. Then the set of all X in P so $AX = BX$ is the line $L \perp AB$ so that $\text{midpt}(A, B) \in L$.

$A * X * B$ and $|AX| = |XB|$.
Exists by Ruler Postulate

WANT

Proof $|AX| = |BX| \Leftrightarrow X \in L$.

Case 1 $X \in AB$. Take ruler f so $f(B) > f(A)$.

Then $|AX| = |BX| \Leftrightarrow |f(B) - f(X)| = |f(A) - f(X)| \Leftrightarrow$ Squares are equal (since quantities ≥ 0) $\Leftrightarrow$

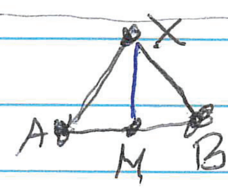
$$f(X)^2 - 2f(B)f(X) + f(B)^2 =$$

$$f(X)^2 - 2f(A)f(X) = f(A)^2 \Leftrightarrow$$

$$f(B)^2 - f(A)^2 = 2f(X)[f(B) - f(A)] \Leftrightarrow \text{(why?)}$$

algebra

$$\boxed{\tfrac{1}{2}\big(f(B) + f(A)\big) = f(X).}$$



M midpt.

Case 2 $X \notin AB$.

If $|XA| = |XB|$, then $\triangle XMA \cong \triangle XMB$ by SSS $\Rightarrow |\measuredangle XMA| = |\measuredangle XMB|$.

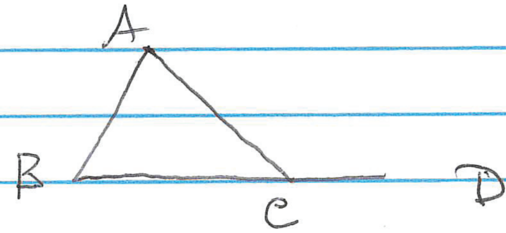
But $A*M*B \Rightarrow 180 = |\measuredangle XMA| + |\measuredangle XMB| = 2|\measuredangle XMA|$

so $|\measuredangle XMA| = 90$.

Conversely, if $|\measuredangle XMA| = 90$ then $\Delta XMA \cong \Delta XMB$ by SAS and hence $|XA| = |XB|$.

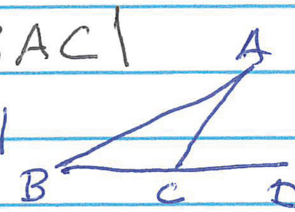
Exterior Angle Theorem (importance often not stressed).

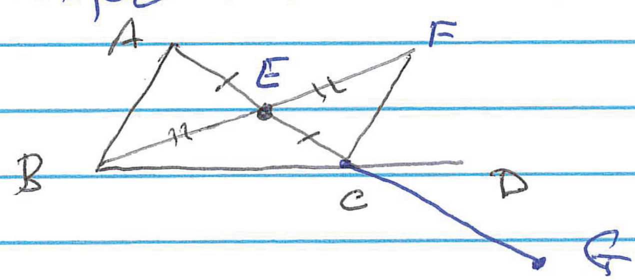
ΔABC & $B*C*D$.



Then $|\measuredangle ACD| > |\measuredangle ABC|, |\measuredangle BAC|$

Can't expect $|\measuredangle ACD| > |\measuredangle ACB|$





$A*C*G$

Proof Construct E, F, G as illustrated:

E = midpt. $[AC]$, $B*E*F$ and $|BF| = 2|BE|$, so $|BE| = |EF|$.

Then $|\measuredangle FEC| = |\angle BEA|$, so

$\Delta EFC \cong \Delta EAB$ by SAS. Hence

$|\measuredangle FCE| = |\measuredangle EAB = \measuredangle BAE|$.

Looks like $|\measuredangle FCE| < |\measuredangle DCE|$. Will follow if $F \in \text{Int} \measuredangle ECD = \measuredangle ACD$.

$B * E * F$ & $A * E * C \Rightarrow F \in A$-side BC.

$B * C * D$ & $B * E * F \Rightarrow$ F, D on op B-side AC, so $F \in D$-side AC

These show $F \in \text{Int} \measuredangle ACD$ & hence

$|\measuredangle ACD| > |\measuredangle ACF = \measuredangle ECF| = |\measuredangle BAC|$.

Must now show $|\measuredangle ACF| > |\measuredangle ABC|$.

Same idea, switch roles ~~of~~ A & B, replace D by G, note $|\measuredangle ACD| = |\measuredangle BCG|$ by Vert. Angle Thm.

Cor. $|\measuredangle ACB| + |\measuredangle BAC| < 180$.

Proof. First show $A \in \text{Int} \measuredangle$ **BCF**.

$A * E * C$ & $B * E * F \Rightarrow$ A & F on same side BC. Also A & B on same side CF.

Hence $180 \geq |\measuredangle BCF| = |\measuredangle BCA| + |\measuredangle ACF| =$

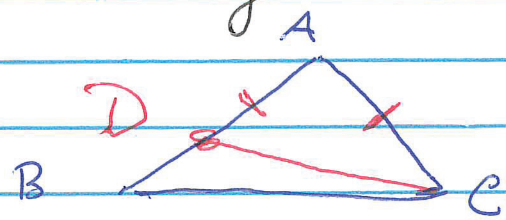
$|\measuredangle BCA| + |\measuredangle BAC|$.

Isosceles $\triangle$ Thm. says that in $\triangle ABC$, $|AB| \neq |AC| \iff |\measuredangle ABC| \neq |\measuredangle ACB|$. In fact

Larger side is opposite larger angle.

$$|AB| > |AC| \implies |\measuredangle ACB| > |\measuredangle ABC|$$

(and conversely — switch roles of B & C).

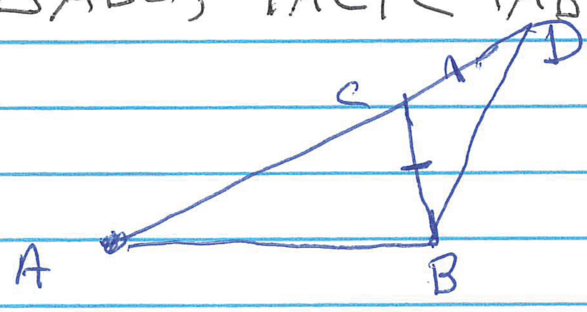


Proof. Have $D \in (AB$ so $|AD| = |AC|$, and in fact $A * D * B$ since $|AD| = |AC| < |AB|$

$|\measuredangle ACB| > |\measuredangle ACD|$ since $D \in \text{Int} \measuredangle ACB$. $|\measuredangle ACD| = |\measuredangle ADC|$ by Isosceles $\triangle$ Thm. $|\measuredangle ADC| > |\measuredangle ABC|$ by Ext. $\measuredangle$ Thm. Combine these to get $|\measuredangle ACB| > |\measuredangle ABC|$.

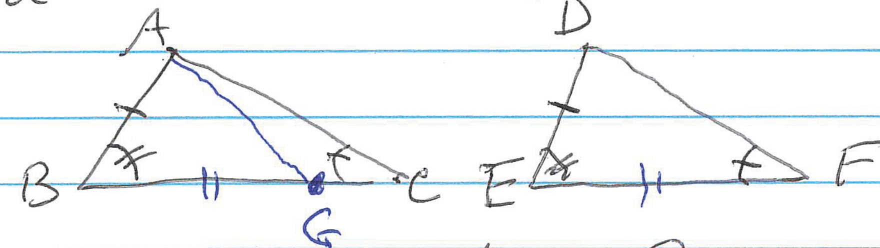
THE TRIANGE INEQUALITY

In $\triangle ABC$, $|AC| < |AB| + |BC|$.



Proof. Locate D so $A*C*D$ and $|CD|=|CB|$. Then $C \in \text{Int}\,\measuredangle ABD$ and $|\measuredangle ADB| = |\measuredangle CBD| < |\measuredangle ADB|$. Hence $|AD| > |AB|$. But $|AD| = |AC|+|CD| = |AB|+|BC|$, so this is greater than $|AC|$.

A.A.S. Congruence In Δs ABC & DEF, if $|\measuredangle ACB| = |\measuredangle DFE|$, $|\measuredangle ABC| = |\measuredangle DEF|$ and $|AB| = |DE|$, then $\Delta ABC \cong \Delta DEF$.



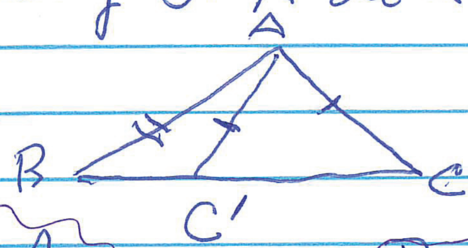
Does not use anything about angle sum of Δ!

Proof Suppose not. Then either $|BC| < |EF|$ or $|EF| < |BC|$. Switching roles of Δs, may as well assume 2nd one. (otherwise $\cong$ by SAS.) Take $G \in (BC$ so $|BG| = |EF|$. Then $|\Delta ABG| \cong |\Delta DEF|$ by SAS, and hence $|\measuredangle AGB| = |\measuredangle DFE| = |\measuredangle ACB|$. But Ext $\measuredangle$ thm implies $|\measuredangle AGB| < |\measuredangle ACB|$. Contradiction. Hence $|BC| = |EF|$.

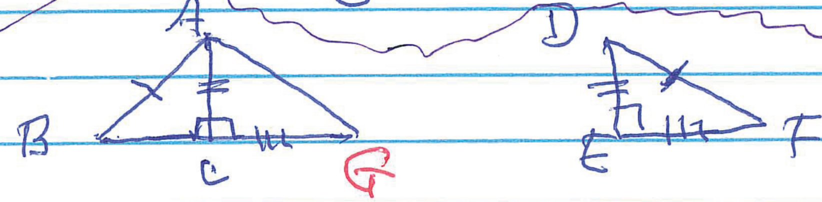
Cor Hypotenuse-angle $\cong$ for rt. $\triangle$s.

Thm. Hypotenuse-side $\cong$ for rt $\triangle$s.

Usually SSA does not imply $\cong$



However, only two noncong $\triangle$s with such data.



Proof Construct G so $B * C * G$ and $|CG| = |EF|$ and $\triangle ACG \cong \triangle DEF$ by SAS.

Then $|DF| = |AG|$ and $|AB| = |DF| \Rightarrow$

$|AB| = |AG| \Rightarrow$

$|\measuredangle ABC| = |\measuredangle AGC|$ by Isosceles $\triangle$ Thm.

But $|\measuredangle AGC| = |\measuredangle DFE|$. Hence $\triangle ABC \cong \triangle DFE$ by A.A.S.

Does not use Pythagorean Theorem.
(as is often the case in H.S. textbooks).

System satisfying

$(\mathcal{P}; \mathcal{L}, d, \alpha)$ or

$(\mathcal{S}; \mathcal{P}, \mathcal{L}, d, \alpha)$ is called

a neutral geometry.