Euclidean parallelism

Def. Two lines L & M ($L \neq M$) are parallel if they are coplanar and disjoint. $L \parallel M$

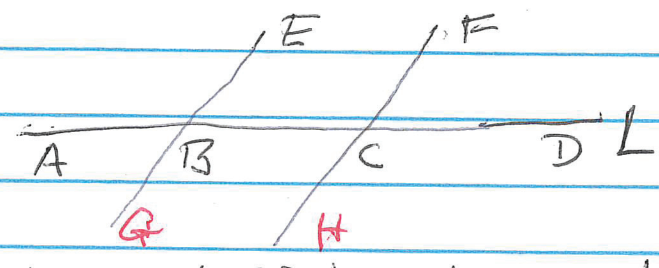
Example $X \notin L$ Let K be $\perp$ to L through X, M = perpendicular to K through X.

Then $L \parallel M$, for if $Y \in L \cap M$ then

(1) $Y \notin K$; otherwise $Y \in K \cap L \cap M$ so both L and M are perpendicular to K at Y CONTRADICTION

(2) there is only one $\perp$ to K through Y.

Even more is true.



E, F same side L.

If $|\measuredangle EBC = \measuredangle EBD| = |\measuredangle FCD|$, then $BE \parallel CF$.

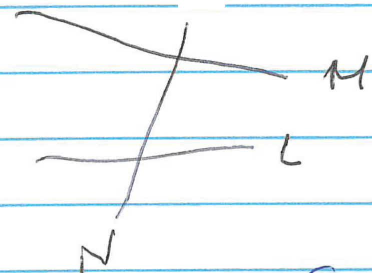
Exclude all options Suppose $X \in BE \cap CF$.

$X \notin L$. Would imply $X = B = C$ since $X \in L \cap BE$ & $X \in L \cap BF$.

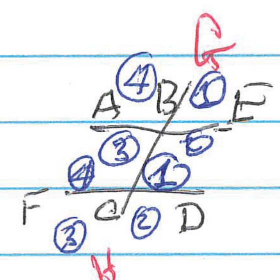
$X \in E \& F$ side of L, for then the consequences of E.A.T. would imply $|\angle FCD| > |\angle EBC|$.
$X \in$ opposite side, for E.A.T. implies
$|\angle FCD| = 180 - |\angle HCD| < 180 - |\angle GBC| =$
$|\angle EBC|$.

Transversals

$L \& M$ coplanar. Transversal N meets both in points, not the point in $L \cap M$.



Angle pairs



A & B & E, D & C & F

A & F on same side BD
D & E on same side BD
A & D on opp sides.

alternate interior angles $\angle ABC$ & $\angle BCD$, $\angle EBC$ & $\angle FCD$
corresponding angles 2 etc.
Consecutive angles $\angle BCD$ & $\angle CBE$ etc.
alternate exterior angles Two pairs

Half of H.S. Geometry Theorem

In prev. settings each of the following implies $AB \parallel CD$.

(1) A pair of corresponding angles have = measures.

(2) Likewise for alternate $\begin{Bmatrix} \text{interior} \\ \text{exterior} \end{Bmatrix}$ angles

(3) A pair of consecutive angles is supplementary (measures add up to 180).

Proofs (1) is the previous result.

(2.1) Will only consider alt. int. pair $\measuredangle ABC$ & $\measuredangle BCD$. Other case follows by switching roles of letters.

Here we are given $|\measuredangle ABC| = |\measuredangle DEF|$. By Vert. $\measuredangle$ Thm., $|\measuredangle GBE| = |\measuredangle ABC| = |\measuredangle DEF|$. Apply (1).

(2.2) Will only consider $\measuredangle ABG$ & $\measuredangle HCD$ as in part 2.1. Supp. Postulate $\Rightarrow$

$|\measuredangle ABG| + |\measuredangle ABC| = 180 = |\measuredangle HCD| + |\measuredangle BCD|$.

We're given $|\measuredangle ABG| = |\measuredangle HCD|$, so $|\measuredangle ABC| = |\measuredangle DEF|$.

Now apply (2.1).

(3) Only consider one case as before. Now we're given $|\measuredangle EBC| + |\measuredangle BCD| = 180$.
Supp. Post. $\Rightarrow |\measuredangle ABC| + |\measuredangle EBC| = 180$.
Hence $|\measuredangle ABC| = 180 - |\measuredangle EBC| = 180 - |\measuredangle BCD|$.
Again apply (2.1).

EUCLIDEAN PARALLELISM

"Fifth" (Parallel) Postulate Given line L and $X \notin L$ there is only one line M so $X \in M$ & $L \parallel M$. (Euclidean Parallel Postulate)

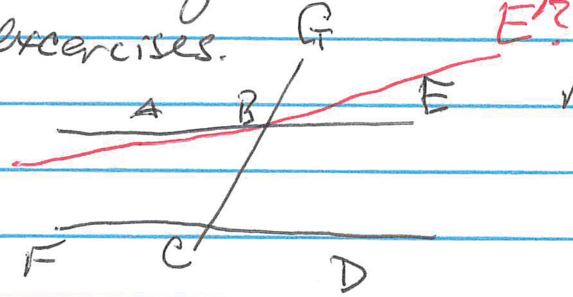
Assume until further notice!

Immediate consequence L, M, N distinct coplanar lines. If $L \parallel M$ & $M \parallel N$, then $L \parallel N$.

Proof If not, then $Y \in L \cap N \Rightarrow L$ & $N \parallel M$ through Y. [Must prove L & N coplanar! (3D version also true but harder to prove)].

Other half of H.S. Theorem. Prev. situation $AB \parallel CD \Rightarrow$ all of the listed sufficient conditions. **(This part requires the || postulate!)**

Proof Only show (1) holds. Others left as excercises.



Assume $AB \parallel CD$.

Protractor Post. $\Rightarrow$ unique ray $[BE'$ so $CBE' \subseteq$ $D+E$ side BC and $|\angle GBE'| = |\overset{\angle BCD}{\angle GCD}|$.

Then AB & BE' are lines through B + $\parallel CD$.

Hence $AB = BE'$ and thus $[BE' = [BE$, so $|\angle GBE| = |\angle GBE'| = |\angle BCD|$.

FILL IN THE REST!!

Cor. 1 Suppose $L, M, N \subseteq$ plane + $L \parallel N$, $L \perp M$. Then also $M \perp N$.

Proof Key point is N & M not parallel (else $L \parallel N$ & $N \parallel M \Rightarrow L \parallel M$, which is false). Now use prev. results.

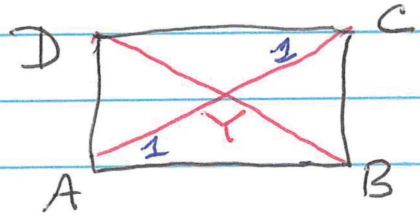
Cor. 2 $L \perp M$, . . . $N \perp K$ and $L \nparallel N$ $\Rightarrow M \nparallel K$.

Proof $M \parallel K$ and $L \perp M \Rightarrow L \perp K$. But $L \perp K$ & $N \perp K$ would $\Rightarrow L \parallel N$, contradiction.

<u>Def</u>: Say A, B, C, D form vertices of a rectangle if $AB \perp BC$, $BC \perp CD$, $CD \perp DA$ and $DA \perp AB$. Automatically convex.

<u>Proposition</u>. $|AB| = |CD|$ and $|AD| = |BC|$.

<u>Proof</u>.



$AB \parallel CD$ & $BC \parallel AD$. Also (AC) & (BD) meet at some point Y. B & D on opp sides $AC \Rightarrow$

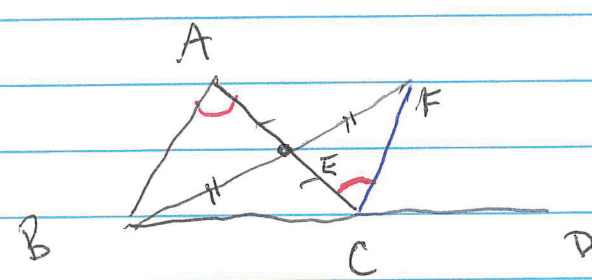
$|\measuredangle DCA| = |\measuredangle ACB|$, also

$|\measuredangle BCA| = |\measuredangle DAC|$. Hence $\triangle ADC \cong \triangle CBA$ by A.S.A., so $|AB| = |CD|$ and $|AD| = |BC|$.

Angle Sum of a Triangle

In $\triangle ABC$, $|\measuredangle BAC| + |\measuredangle ABC| + |\measuredangle ACB| = 180$.

<u>Proof</u>. Use setting of Exterior Angle Thm.



Recall some conclusions

What more can we say if $\parallel$ postulate is true?

$|\measuredangle BAC| = |\measuredangle ACF|$ so $BA \parallel CF$ by first half of thm on transversals. Now EPP $\Rightarrow$ for corresponding angles we have

$|\measuredangle ABC| = |\measuredangle FCD|$.

We saw $F \in \mathrm{Int}\, \measuredangle ACD$, so

$$|\measuredangle ACD| = |\measuredangle ACF| + |\measuredangle FCD| = |\measuredangle BAC| + |\measuredangle ABC|.$$

Hence $180 = |\measuredangle ACB| + |\measuredangle ACD| =$

$$|\measuredangle ACB| + |\measuredangle BAC| + |\measuredangle ABC|.$$