Math 133
Fall 2021

SOLUTIONS FOR WEEK 03 EXERCISES

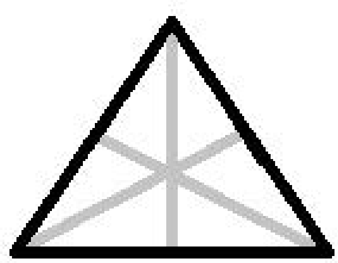
Assume that $(\mathbf{S}; \mathcal{P}; \mathcal{L}; d; \alpha)$ or $(\mathbf{P}; \mathcal{L}; d; \alpha)$ is a system which satisfies the Incidence, Ruler, Plane or Space Separation (as appropriate), Angle Measurement and Triangle Congruence Axioms; in other words, the system is a **neutral geometry**.

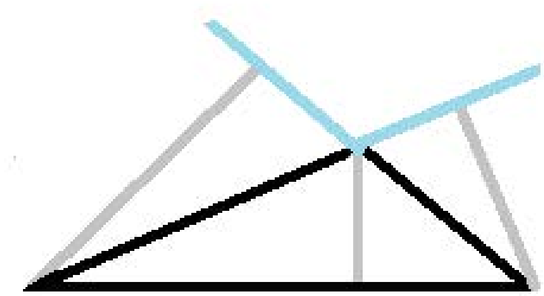
1. We know that D cannot be equal to either A or C, because this would imply that either $|\angle BCA|$ or $|\angle CAB|$ would be equal to $90°$. Thus one of the three points A, C, D must be between the other two. If we have $A * C * D$, then the Exterior Angle Theorem would imply that $|\angle ACB| > |\angle CDB| = 90°$, which would contradict our assumption that $|\angle ACB| = |\angle BCA| < 90°$. Similarly, if we have $D * A * C$, then the Exterior Angle Theorem would imply that $|\angle BAC| > |\angle BDA| = 90°$, which would contradict our assumption that $|\angle CAB| = |\angle BAC| < 90°$. The only remaining possibility for the collinear points A, B, D is the betweenness relation $A * D * C$.■

2. The first part follows from the preceding exercise because the measures of at least two vertex angles are less than $90°$ (by the Exterior Angle Theorem).

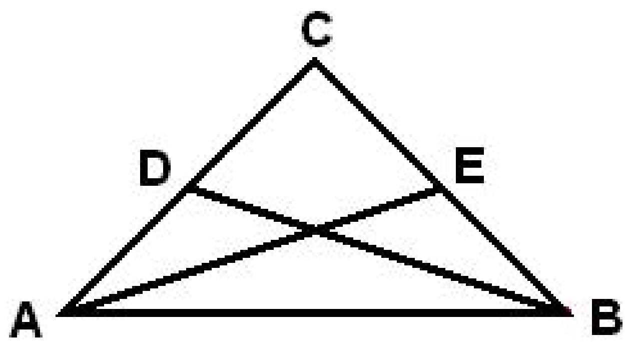
If all three vertex angles are acute (measure $< 90°$), then the feet of all three perpendiculars lie on the respective open segments. However, if $\angle ABC$ in $\triangle ABC$ is a right angle, then The feet of the perpendiculars from A and C are the point B, which does not lie on (AB) or BC. Likewise, if $\angle ABC$ in $\triangle ABC$ is an obtuse angle (measure $> 90°$), then the Exterior Angle Theorem implies that the feet Y and Z of the perpendiculars from A and C must satisfy $Y * B * C$ and $A * B * Z$ respectively.■

Hee are sketches of an example where all three statements are true and an example where only one statement is true.■



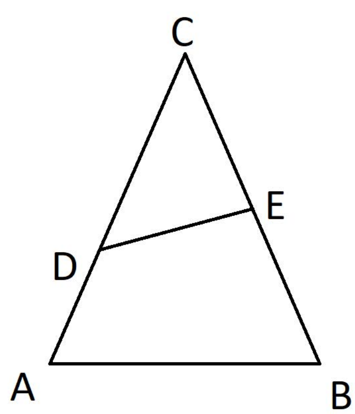


3. (a) Here is a drawing:



Note first that $|\angle DBE| = |\angle DBA| = \frac{1}{2}|\angle ABC| = \frac{1}{2}|\angle DBC| = |\angle DBF|$. Since $|BD| = |BD|$ trivially holds and we are given $|\angle DEB| = |\angle DFB|$, we have $\triangle DEB \cong \triangle DFB$ by AAS, and therefore we also have $|DE| = |DF|$.$\blacksquare$

(b) Here is a drawing:



Since the larger angle is opposite the longer side, we have $|\angle CDE| > |\angle CED|$. Therefore $|\angle BED| = 180 - |\angle CED| > 180 - |\angle CDE| = |\angle ADE|$.$\blacksquare$

4. The points X, Y, Z are not collinear because a line cannot intersect all three open sides of a triangle. Also, the betweenness hypotheses imply

$$|AB| = |AX| + |XB|,$$

$$|BC| = |BY| + |YC|, \text{ and}$$

$$|AC| = |AZ| + |ZC|.$$

Finally, the Triangle Inequality implies that

$$|XY| < |BX| + |BY|,$$

$$|YZ| < |CY| + |CZ|, \text{ and}$$

$$|XZ| < |AX| + |AZ|.$$

If we add these we obtain

$$|XY| \ + \ |YZ| \ + \ |XZ| \quad < \quad |BX| \ + \ |BY| \ + \ |CY| + |CZ| \ + \ |AX| \ + \ |AZ|$$

and using the betweennes identities we see that the right hand side is equal to $|AB|+|BC|+|AC|$; thus we have shown the inequality stated in the exercise.■

5. Let X and Y be points of P, and let $L = XY$. Take M to be the unique perpendicular to L at X. If N is a line through X which is perpendicular to L, by uniqueness we have $N = M$. Therefore there cannot be a line N in P through X which is perpendicular to both L and M.■

6. (a) If there is a point $E \in (AC \cap (BD$, then $\angle CAB = \angle EAB$ and $\angle DBA = \angle EBA$, so by the Exterior Angle Theorem we know that

$$|\angle CAB| + |\angle DBA| \quad = \quad |\angle EAB| + |\angle EBA| \quad < \quad 180$$

which is what we wanted to prove.■

(b) In this case the two lines meet at a point E such that $E * A * C$ and $E * B * D$. By the reasoning in (a) we know that $|\angle EAB| + \angle EBA| < 180$ and by the Supplement Postulate this translates to

$$|\angle CAB| + |\angle DBA| \quad = \quad (180 - |\angle EAB|) + (180 - |\angle EBA|) \quad =$$

$$360 - (|\angle EAB| + \angle EBA|) \quad > \quad 360 - 180 \quad = \quad 180$$

which is what we wanted to prove.■