Math 133
Fall 2021

MORE EXERCISES FOR WEEK 03

For the first exercise assume that $(\mathbf{S}; \mathcal{P}; \mathcal{L}; d; \alpha)$ or $(\mathbf{P}; \mathcal{L}; d; \alpha)$ is a system which satisfies the axioms for a *neutral geometry*, and for the remaining exercises assume that the system also satisfies the Euclidean Parallel Postulate (*i.e.*, the axioms for a *Euclidean geometry*.

7. If L and M are parallel lines, then by a previous exercise we know that all points of L lie on the same side of M and all points of M lie on the same side of L. Define the **strip between** L **and** M to be the set of points x such that x and L are the same side of M, and x and M are the same side of L. Prove that the strip between L and M is convex and nonempty. Specifically, prove that if $A \in L$ and $B \in M$, then the midpoint C of (AB) lies in this set.

8. Suppose that $\angle ACB$ in the plane is inscribed in a semicircle; in other words, if X is the midpoint of the segment $[AB]$ then all three points A, B, C are equidistant from X. Then $\angle ACB$ is a right angle. Conversely, show that if $\angle ACB$ is a right angle then the midpoint X is equidistant from A, B and C.

9. (*The other half of Euclid's Fifth Postulate*) Let AB be a line, and let C and D be points such that A, B, C and D are coplanar and both C and D lie on the same side of AB. Prove the following:

(a) The open rays $(AC$ and $(BD$ have a point in common if $|\angle CAB| + |\angle DBA| < 180°$.

(b) The lines AC and BD meet at a point on the side of AB which is opposite the side containing C and D if $|\angle CAB| + |\angle DBA| > 180°$.

10. (*Strengthened Exterior Angle Theorem*) Given $\triangle ABC$, let D be a point such that $B * C * D$. Then we have $|\angle ACD| = |\angle ABC| + |\angle BAC|$.

11. (*Third Angles Are Equal Theorem*) Suppose we have two ordered triples of noncollinear points (A, B, C) and (D, E, F) satisfying $|\angle ABC| = |\angle DEF|$ and $|\angle CAB| = |\angle FDE|$. Then we also have $|\angle ACB| = |\angle DFE|$.

12. Prove that an isosceles triangle $\triangle ABC$ is equilateral if and only if (at least) one of the angle measurements $|\angle ABC|$, $|\angle BCA|$ or $|\angle CAB|$ is equal to $60°$, and in this case **ALL** of the angle measurements above are equal to $60°$.

13. Suppose that A, B, C and D (in that order) form the vertices of a parallelogram. Then we have $|\angle ADC| = |\angle CDA|$, $|AB| = |CD|$, $|AD| = |BC|$, and $|\angle BCD| = |\angle DAB|$. Also prove that

$$|\angle ADC| \quad = \quad |\angle ABC| \quad = \quad 180° - |\angle DAB| \quad = \quad 180° - |\angle DCB| \ .$$

14. Let L and M be parallel lines. Let X be a point on one of these lines, let Y be a point of the other line such that XY is perpendicular to L and M, let Z be another point on one of these lines, and let W be a point of the other line such that ZW is perpendicular to L and M. Then we have $|XY| = |ZW|$. — In everyday language, *two parallel lines are everywhere equidistant*. The common value of the numbers $|XY|$, $|ZW|$, etc. is frequently called the **distance between** L **and** M.

15. Suppose we are given isosceles triangle $\triangle ABC$ with $|\angle ABC| = |\angle ACB|$, let D satisfy $B * A * D$, and suppose that $E \in \text{Int } \angle DAC$ such that $[AE$ bisects $\angle DAC$. Prove that AE is parallel to BC. [*Hint:* Draw a picture.]