Math 133
Fall 2021

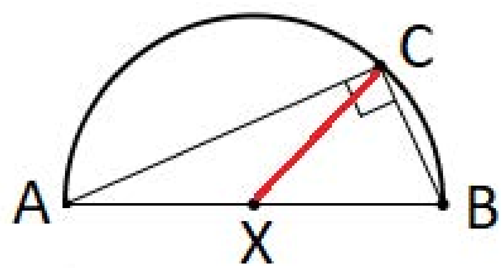
MORE SOLUTIONS FOR WEEK 03 EXERCISES

For the first exercise assume that $(\mathbf{S}; \mathcal{P}; \mathcal{L}; d; \alpha)$ or $(\mathbf{P}; \mathcal{L}; d; \alpha)$ is a system which satisfies the axioms for a *neutral geometry*, and for the remaining exercises assume that the system also satisfies the Euclidean Parallel Postulate (*i.e.*, the axioms for a *Euclidean geometry*).

7. By construction the strip is the intersection of two half planes; aince both are convex, their intersection, which is the strip, must also be convex.

To show the set is nonempty, it suffices to prove the assertion in the final sentence. Since the midpoint satisfies $A * C * B$ where $A \in L$ and $B \in M$, it follows that C and B (hence all of M) lie on the same side of L, and also that C and A (hence all of L) lie on the same side of M.■

8. Here is a drawing:



Assume that X is the midpoint of the segment $[AB]$ and $|AX| = |BX| = |CX|$. Then $\triangle AXC$ and $\triangle BXC$ are isosceles triangles, and therefore $|\angle BAC = \angle XAC| = |\angle XCA|$ and $|\angle ABC = \angle XBC| = |\angle XCB|$; by the preceding paragrphaph, it follows that $|\angle XCA| + |\angle XCB| = 90°$. Furthermore, we know that $C \in (AB)$, so that $X \in \text{Int } \angle ACB$ and hence $|\angle ACB| = |\angle XCA| + |\angle XCB| = 90°$. Therefore the vertex angle at C is a right angle.■

9. (a) Let E be a point on the same side of AB as C and D such that $|\angle CAB| + |\angle XBA| < 180° - |\angle CAB|$. Then $DB \neq XB$ because $|\angle DBA| < 180° - |\angle CAB| - |\angle XAB|$. By a previous exercise we know that XB is parallel to AB, and therefore the Euclidean Parallel Postulate implies that the lines DB and AC have a point in common; note that these two lines are unequal because their intersections with AB are B and A respectively. We need to show that the intersection point of the two lines lies in the same side of AB as X, C and D.

Suppose this is not the case then the common point Y lies on the other side of AB; we then have $Y * A * C$ and $Y * B * D$. By the Exterior Angle Theorem we must have $|\angle YAB| + |\angle YBA| < 180°$. But now the Supplement Postuulate implies that $|\angle CAB| = 180° - |\angle YAB|$ and $|\angle DBA| = 180° - |\angle YBA|$. As in the argument for the first half of Euclid's Fifth Postulate, these three relations impliy that $|\angle CAB| + |\angle DBA| > 180°$, a contradiction. The source of this contradiction is the assumption in the first sentence of the paragraph, and therefore we see that the two open rays must have a point in common.■

(b) This is similar to (a). Once again the two lines must have a common point W, and it is not on AB. If it were on the same side as C and D, the reasoning in the preceding paragraph would imply $|\angle CAB = \angle WAB| + |\angle WBA = \angle DBA| < 180°$, and this contradiction means that the intersection must lie on the other side of AB.■.

10. The Supplement Postulate implies that $|\angle ACB| + |\angle ACD| = 180°$. Since the angle sum of a triangle is 180° we then have

$$|\angle BAC| + |\angle ABC| \quad = \quad 180° - |\angle ACB| \quad = \quad |\angle ACD|$$

which is what we wanted to prove.■

11. Since the angle sum of a triangle is 180° we have

$$|\angle ABC| + |\angle CAB| + |\angle ACB| = 180° = |\angle DEF| + |\angle FDE| + |\angle ACB| = |\angle DFE|$$

so if we subtract the equal quantities $|\angle ABC| + |\angle CAB|$ and $|\angle DEF| + |\angle FDE|$ from the left and right hand sides respectively, we conclude that $|\angle ACB| = |\angle DFE|$.■

12. Since an equilateral triangle is equiangular, we have

$$|\angle ABC| + |\angle BCA| + |\angle CAB| \ = \ 180° \ = \ 3|\angle ABC| \ = \ 3|\angle BCA| \ = \ 3|\angle CAB|$$

and hence $|\angle ABC| = |\angle BCA| = |\angle CAB| = 60°$.

Conversely, assume that one of the vertex angle measurements is 60°. It will suffice to consider the case where the isosceles triangle satisfies $|AB| = |AC|$; the other cases ($|BA| = |BC|$ and $|CA| = |CB|$) will follow if we interchange the roles of A, B and C in a suitable fashion.

In the selected case we have $|\angle ABC| = |\angle ACB|$. If their common angle measure is 60°, then we have

$$|\angle BAC| \quad = \quad 180° - |\angle ABC| - |\angle ACB| \quad = \quad 180° - 2|\angle ABC| \quad = \quad 180° - 120° \quad = \quad 60°$$

so that $\triangle ABC$ is equiangular and hence equilateral.

Finally, suppose that $|\angle BAC| = 60°$. Since the triangle is isosceles, the argument in the preceding paragraph shows that

$$120° \quad = \quad |\angle ABC| + |\angle ACB| 60° \quad = \quad 2|\angle ABC| \quad = \quad 2|\angle ACB|$$

and hence $|\angle ABC| = |\angle ACB| = 60°$. Therefore $\triangle ABC$ is equiangular and hence equilateral.■

13. We know that a parallelogram is a convex quadrilateral, and hence the diagonal segments (BD) and (AC) meet a some point X. It follows that B and D lie on opposite sides of AC, and similarly A and C lie on opposite sides of BD. Therefore $\{\angle DCA, \angle CAB\}$ and $\{\angle DAC, \angle ACB\}$ are pairs of alternate interior angles. By ASA we then have $\triangle BAC \cong \triangle DCA$. In particular, this implies that $|\angle ADC| = |\angle CDA|$, $|AB| = |CD|$, and $|AD| = |BC|$. The assertion that $|\angle BCD| = |\angle CAB|$ can be proven by cyclically interchanging the roles of the vertices in the preceding argument; specifically, we let B, C, D and A take the roles of A, B, C and D respectively.

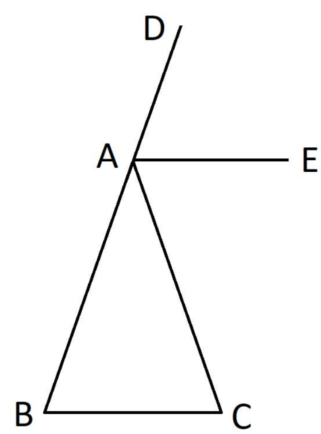
To prove the statements about supplementary angles in the final sentence of the exercise, first let E be a point such that $A * D * E$. Then by the results on corresponding angles and the Supplement Postulate we know that $|\angle DAB| = |\angle EDC| = 180^\circ - |\angle ADC|$ and the remaining conclusions follow from this equation and the results of the preceding paragraph.■

14. Without loss of generality, we may as well assume that X lies on L; the proof in the case $X \in M$ follows by reversing the roles of L and M in the argument which follows. Since $X \in L$ we also must have $Y \in M$. There are now a few separate cases. Let us dispose of the case where $Z = Y$ first. In this situation we also have $W = X$ and hence the distance equation is a triviality.

Suppose next that Z lies on L and is not equal to X; we claim that W is also not equal to Y, for if $W = Y$ then by uniqueness of perpendiculars to a line at a point we could conclude that X, Y and Z were collinear. This is impossible because the collinearity relationship would mean that the line XZ is perpendicular to M, while the hypothesis implies that $L = XZ$ is parallel to M. Since two lines perpendicular to a third line are parallel, it follows that XY is parallel to ZW, and hence X, Y, W and Z (in that order) form the vertices of a parallelogram. Therefore the exercise on parallelograms implies that $|XZ| = |YW|$.

Suppose now that Z lies on M; by an earlier part of the argument we know the result holds if $Z = Y$, so suppose now that they are distinct. We shall apply the reasoning of the previous paragraph systematically. First of all, if W is the point on L such that ZW is perpendicular to L and M, then this reasoning implies that W is not equal to X. It follows now that XZ is parallel to YW. Therefore X, Z, Y and W (in that order) form the vertices of a parallelogram. As before, the basic result on parallelograms implies that $|XZ| = |YW|$.■

15. Here is a drawing for the problem:



By construction E lies in the interior of $\angle DAC$, so that E and C lie on the same side of AB. We then have the following chain of equations:

$$2|\angle ABC| = |\angle ABC| + |\angle ACB| = 180^\circ - |\angle BAC| = |\angle DAC| = 2|\angle DAE|$$

(isosceles triangle, angle sum of triangle is 180°, supplementary angles, bisection hypothesis) so that $|\angle ABC| = |\angle DAE|$. Furthermore, by construction $\angle DAE$ and $\angle ABC$ are corresponding angles with respect to the transversal AB. Therefore the lines AE and BC must be parallel.■