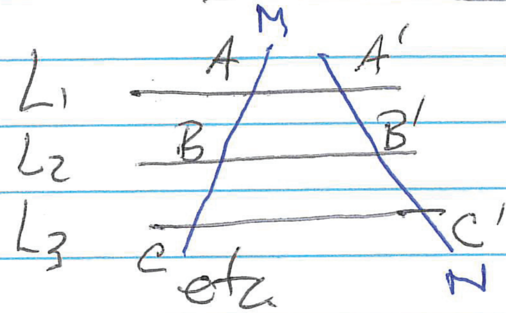
Parallel Projection



L_i mutually $\parallel$

M & N transversals

First observation $A*B*C \Rightarrow A'*B'*C'$

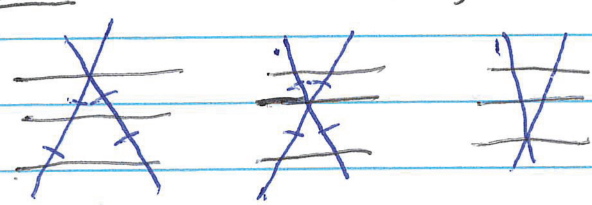
Proof B lies in strip between L_1 & L_3 (see HW).

$B \in L_3$-side $L_1 \cap L_1$-side L_3 (def of strip)

Either $B'=B$ or $BB'=L_2$. In either case, $B' \in$ [previous] $\Rightarrow A'*B'*C'$.

Prop 1 $|AB|=|AC| \Rightarrow |A'B'|=|A'C'|$.

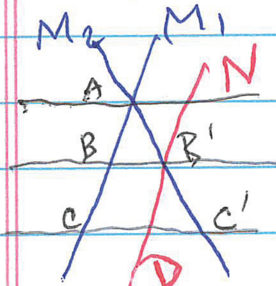
Proof Cases $A=A'$, $B=B'$ or $C=C'$



1st & 3rd same up to switching roles of variables $A \leftrightarrow C$, $M_1 \leftrightarrow M_2$ etc.

Main case $M_1 \nparallel M_2$

Do $M_1 \parallel M_2$ at end.



$A=A'$

Take N through B' $N \parallel M_1$

$|\measuredangle ABB'| = |\measuredangle ACC'|$ (Corresp $\measuredangle$s)

$|\measuredangle AB'B| = |\measuredangle AC'C|$

$D \in N \cap L_3$

A & C' on opp sides N

A & C on same side N

$\Rightarrow$ C & C' on opp sides N.

hence $C*D*C'$

Also recall $|\angle AB'B| = |\angle AC'C|$ by remarks on prev. page.

This implie B, B', D, C vertices of ▱, and $\measuredangle ACC' = \measuredangle ACD$ and $\measuredangle B'DC'$ are corresp $\measuredangle$s.

Hence $|AB| = |B'D|$ ($= |BC|$) and

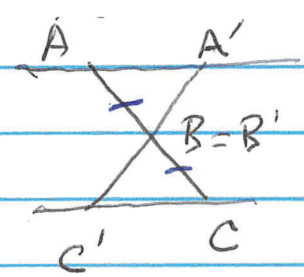
$|\measuredangle ACC'| = |\measuredangle ABB'| = |\measuredangle C'B'D|$. Hence

(asa)

$\triangle ABB' \cong \triangle B'DC'$, so that

$|A'B'| = |B'C'|$.

Case $B = B'$



$|AB| = |BC|$ given

$|\measuredangle ABA'| = |\measuredangle CBC'|$ Vertical angles

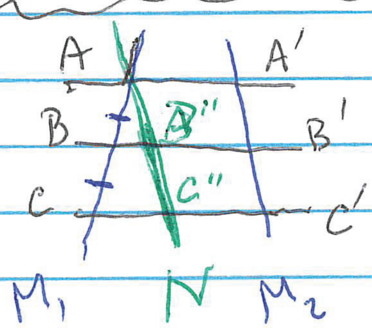
$|\measuredangle AA'C| = |\measuredangle CC'A|$ Alternate Interior Angles.

So $\triangle ABA' \cong \triangle CBC'$ by ASA, and hence

$|A'B = A'B'| = |C'B = C'B'|$.

Case $\{A, B, C\} \cap \{A', B', C'\} = \emptyset$.

{ no condition whether $M_1 \parallel M_2$ or NO.

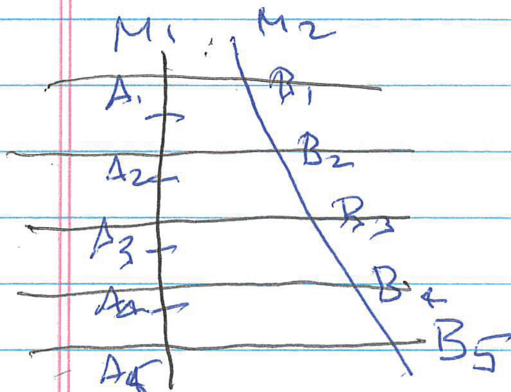


Choose $N \parallel M_2$ so $A \in N$

Then $N \neq M_2$ otherwise $A \in M_2$, which is not the case

Previous reasoning shows $|AB''| = |B''C'|$. But now we have $\square$s $AA'B''B$ + $B''B'C''C$, so $|AB''| = |A'B'|$, $|B''C''| = |B'C'|$ and finally $|A'B'| = |AB''| = |B'C''| = |B'C'|$.

<u>Notebook Paper Theorem</u> L_i, M_j (mutually $\parallel$) as before with $A_i \in L_i \cap M_1$, $A_1 * A_2 * A_3 \cdots * A_m$. $B_i \in L_i \cap M_2$. Then $B_1 * \cdots * B_m$. Furthermore, if $|A_{i+1} A_i| = |A_2 A_1|$ for all i, then $|B_{i-1} B_i| = |B_2 B_1|$ for all i.

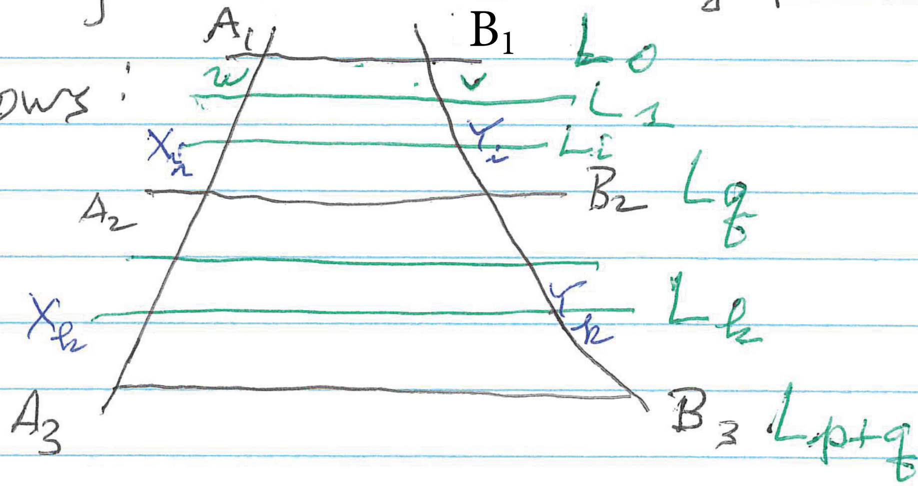


Apply preceding successively to L_{i-1}, L_i, L_{i+1}

<u>Rational Proportionality Thm.</u> $L_1\ L_2\ L_3$ as before $A_i \in M_1 \cap L_i$, $B_i \in M_2 \cap L_i$. If $\dfrac{|A_2 A_3|}{|A_1 A_2|}$ is rational, then $\dfrac{|B_2 B_3|}{|B_1 B_2|} = \dfrac{|A_2 A_3|}{|A_1 A_2|}$.

By algebra we also have $\dfrac{|B_1B_2|}{|A_1A_2|} = \dfrac{|B_2B_3|}{|A_2A_3|}$.

Proof Let $\dfrac{|A_2A_3|}{|A_1A_2|} = \dfrac{p}{q}$, so $\dfrac{|A_2A_3|}{p} = \dfrac{|A_1A_2|}{q} =$ w ("unit length"). Draw auxiliary parallels L_i as follows:



$\boxed{|X_{i+1}X_i| = w \text{ all } i}$

Previous results show that $|Y_{i+1}Y_i| = v$ all i, so also by betweenness

$$\left.\begin{array}{ll} |A_1A_2| = qw, & |A_2A_3| = pw \\ |B_1B_2| = qv, & |B_2B_3| = pv, \end{array}\right\} \text{ so } \frac{|B_2B_3|}{|B_1B_2|} = \frac{p}{q}.$$

Irrational $\dfrac{|A_2A_3|}{|A_1A_2|}$. Need more sophisticated ideas, took Greeks about 200 years to find a solution.

<u>Condition of Eudoxus</u> $0 < x, y \in \mathbb{R}$.
Then $x = y \iff$
(a) every positive rational $\frac{p}{q} < x$ satisfies $\frac{p}{q} < y$.
(b) Same with inequality directions reversed.
<u>Key idea</u> If $0 < x < y$ there is some $\frac{p}{q}$ so $x < \frac{p}{q} < y$. (Look at decimal expansions, for example).

$\boxed{\text{See Moise 11.3–11.4 for details}}$

- read & understand passively!

<u>Similar Triangles</u> $\triangle ABC \underset{r}{\sim} \triangle DEF$ Vertices ordered
(Similar, with ratio of similitude $= r$) $\iff$

$|\measuredangle ABC| = |\measuredangle DEF|$, $|\measuredangle BCA| = |\measuredangle DFE|$, $|\measuredangle CAB| = |\measuredangle FDE|$

and
$$\frac{|DE|}{|AB|} = \frac{|DF|}{|AC|} = \frac{|EF|}{|BC|} = r \left(\begin{matrix}\text{some}\\ r\end{matrix}\right).$$

<u>Abstract stuff.</u> $\triangle ABC \cong \triangle DEF \iff \triangle ABC \underset{1}{\cong} \triangle DEF$

$\triangle ABC \underset{1}{\sim} \triangle ABC$, $\triangle ABC \underset{r}{\sim} \triangle DEF \Rightarrow \triangle DEF \underset{1/r}{\sim} \triangle ABC$

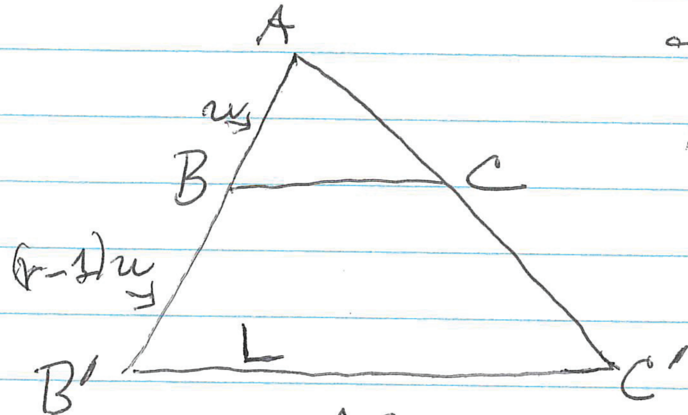
$\triangle ABC \underset{r}{\sim} \triangle DEF$ & $\triangle DEF \underset{s}{\sim} \triangle GHK \Rightarrow \triangle ABC \underset{rs}{\sim} \triangle GHK$

AA Similarity Theorem Given $\triangle ABC$ & $\triangle DEF$ s.t. $|\measuredangle BAC| = |\measuredangle EDF|$, $|\measuredangle ABC| = |\measuredangle DEF|$.
Then $\triangle ABC \sim \triangle DEF$.

Proof. We also have $|\measuredangle ACD| = |\measuredangle DFE|$.

Let $r = \frac{|DE|}{|AB|}$.

Now suppose $r \leq 1$. Switching the roles of the triangles if nec., can assume $r \geq 1$.



Take $B' \in (AB$ so that $|AB'| = r|AB|$. Let line L through B' so that $L \parallel BC$. Notice $A*B*B'$ ($r>1$)

CLAIM L & AC not parallel; otherwise

$L \parallel AC$ & $L \parallel BC \Rightarrow BC \parallel AC$, false since these meet at C. Suppose L & AC meet at C'. Then $L \subseteq B'$-side AC & $A*B*B' \Rightarrow C'$ & A on opp sides, So C, C' on same side and $A * C * C'$. By prev. thm.

$$\frac{|CC'|}{|AC|} = \frac{|BB'|}{|AB|} = r-1, \text{ so } \frac{|AB'|}{|AB|} \underset{A*B*B'}{=} \frac{|AB|+|BB'|}{|AB|}$$

$$= \frac{|BB'|}{|AB|} + 1; \text{ likewise } \frac{|AC'|}{|AC|} = \frac{|CC'|}{|AC|} + 1. \text{ Since}$$

$$\frac{|CC'|}{|AC|} = \frac{|BB'|}{|AB|}, \text{ get } r = \frac{|AB'|}{|AB|} = \frac{|AC'|}{|AC|}.$$

Also $\measuredangle BAC = \measuredangle B'AC'$, $|\measuredangle ABC| = |\measuredangle AB'C'|$, $|\measuredangle ACB| = |\measuredangle AC'B'|$. Combining with hypotheses, get $|\measuredangle B'AC'| = |\measuredangle EDF|$, $|\measuredangle AB'C'| = |\measuredangle DEF|$, and

$|DE| = r|AB| = |AB'|$, so $\triangle AB'C' \cong \triangle DEF$ by SAS.

Will suffice to prove $\triangle ABC \sim \triangle DEF$. Have shown

AAA & $\dfrac{|AB'|}{|AB|} = \dfrac{|AC'|}{|AC|}$. If we substitute $A \to B^*$, $B \to C^*$, $C \to A^*$,

equiv to $\dfrac{|DE|}{|AB|} = \dfrac{|DF|}{|AC|}$

$D \to E'$, $E \to F'$, $F \to D'$. Then $\dfrac{|D^*F^*|}{|A^*C^*|} = \dfrac{|D^*E^*|}{|A^*B^*|}$ or $\dfrac{|EF|}{|BC|} = \dfrac{|DF|}{|AC|}$.

Hence $\triangle ABC \sim \triangle AB'C' \cong \triangle DEF$.

Construction yields

<u>Cor.</u> Given $\triangle ABC$ & $r \geq 1$. Then can find $\triangle AB'C'$ so $B' \in (AB$, $C' \in (AC$, $\triangle ABC \underset{r}{\sim} \triangle AB'C'$.