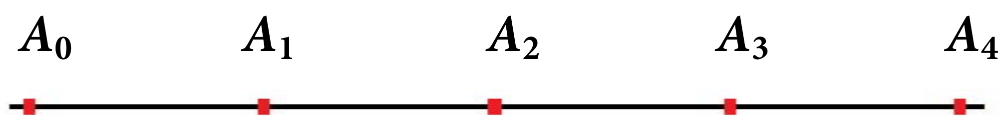
Math 133
Spring 2022
L07A

HIGHER ORDER BETWEENNESS RELATIONSHIPS

In Lecture 07 the notion of betweenness was extended to sets of more than 3 collinear points. The purpose of this document is to formalize this concept and establish some of its basic properties, particularly as they are needed to discuss the Notebook PaperTheorem, proportion and similarity.

Definition. Let $A_0, \cdots, A_n$ be a set of distinct collinear points where $n \geq 2$. These points are said to be in the order $A_0 * \cdots * A_n$ if and only if for each triple (i, j, k) such that $0 \leq i < j < k \leq n$ we have the usual betweenness relation $A_i * A_j * A_k$.



It is not difficult to see that every set of $n+1 \geq 3$ points can be placed in some order.

PROPOSITION. *Given a set of $n+1 \geq 3$ collinear points $A_0, \cdots, A_n$ there is a reordering (permutation) $\sigma(0), \cdots, \sigma(n)$ of $\{0, \cdots, n\}$ so that $A_{\sigma(0)} * \cdots * A_{\sigma(n)}$ is true.*

Proof. Let L be the line containing the points, let $fL \to \mathbb{R}$ be a ruler function for L and let $x_i = f(A_i)$. Then there is some reordering (permutation) $\sigma(0), \cdots, \sigma(n)$ of $\{0, \cdots, n\}$ so that $x_{\sigma(0)} < \cdots < x_{\sigma(n)}$ is true. For each for each triple $(\,(\sigma(i), \sigma(j), \sigma(k)\,)$ such that $0 \leq \sigma(i) < \sigma(j) < \sigma(k) \leq n$ we then have

$$0 \;<\; |x_{\sigma(k)} \;-\; x_{\sigma(i)}| \;=\; |x_{\sigma(j)} \;-\; x_{\sigma(i)}| \;+\; |x_{\sigma(k)} \;-\; x_{\sigma(j)}|$$

and therefore $A_{\sigma(i)} * A_{\sigma(j)} * A_{\sigma(k)}$ is true. By definition, it follows that $A_{\sigma(0)} * \cdots * A_{\sigma(n)}$ is true.■

For large values of n we need a criterion for recognizing when $A_0, \cdots, A_n$ are in the order $A_0 * \cdots * A_n$:

THEOREM. *Let L be a line, and let let $A_0, \cdots, A_n$ be a set of distinct points on L, and let $f : L \to \mathbb{R}$ be a ruler function for L such that $f(A_0) < f(A_i)$ for some $i > 0$. Then the points are in the order $A_0 * \cdots * A_n$ if and only if $f(A_0) < f(A_1) \cdots < f(A_n)$.*

Proof. If the chain of inequalities holds, then the points are in order by the argument in the previous proposition. Conversely, suppose that the points are in the order $A_0 * \cdots * A_n$ and only the single inequality $f(A_0) < f(A_i)$ is known. If $n = 2$ the proof of this converse is a consequence of the earlier result on recognizing betweenness in terms of ruler functions: Betweenness implies that either $f(A_0) < f(A_1) < f(A_2)$ or $f(A_0) > f(A_1) > f(A_2)$, and since we know that either $f(A_0) < f(A_1)$ or $f(A_0) < f(A_2)$, the first alternative must be true.

We can now proceed by finite induction. If $\mathbf{P}_m$ is the statement of the proposition fpr $n = m$, we know that $\mathbf{P}_2$ is true. Assume now that $\mathbf{P}_{n-1}$ is true for $n \geq 3$. The proof that $\mathbf{P}_n$ is true now splits into two cases.

Case 1. $f(A_0) < f(A_i)$ where $i < n$. Since $\mathbf{P}_{n-1}$ is assumed to be true we have $f(A_0) < f(A_1) \ \cdots \ < f(A_{n-1})$. We know that $\mathbf{P}_2$ is also true, and this yields the inequalities $f(A_{n-2}) < f(A_{n-1}) < f(A_n)$. If we combine these inequalities we obtain the desired string of inequalities $f(A_0) < \ \cdots \ < f(A_{n-1}) < f(A_n)$.

Case 2. $f(A_0) < f(A_n)$. Since $\mathbf{P}_2$ is true and $A_0 * \ \cdots \ * A_n$ implies $A_0 * A_{n-1} * A_n$ we also have $f(A_0) < f(A_{n-1}) < f(A_n)$. Therefore we are again in Case 1 where $i = n - 1$, and accordingly we can complete the proof in this case using the argument for that case.■

Example. If we are given a ray $[AB$ and a distance $d > 0$, then we can find points $X_1, \ \cdots \ , X_n \in [AB$ so that $|AX_k| = k\,d$ for $1 \leq k \leq n$. It follows that we have the betweenness relation $A * X_1 * \cdots * X_n$.