Basic similarity theorems

AA similarity - done

SAS similarity Given ΔABC & ΔDEF so $\frac{|AB|}{|BC|} = \frac{|DE|}{|EF|}$ & $|\measuredangle ABC| = |\measuredangle DEF|$, then

$\Delta ABC \sim \Delta DEF$.

Proof. Have $\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|} = \frac{1}{r}$. May as well assume $r \geq 1$ (reverse roles of ΔABC & ΔDEF if ratio ≤ 1). Use result at end of L7 to find ΔGHK so $\Delta ABC \sim \Delta GHK$ with ratio r. Then $|GH| = r|AB| = |DE|$

$|HK| = r|BC| = |EF|$ + $|\measuredangle GHK| = |\measuredangle ABC| = |\measuredangle DEF|$

$\Rightarrow \Delta GHK \cong \Delta DEF$ by SAS congruence.

Since $\Delta ABC \sim \Delta GHK \cong \Delta DEF$, conclude

$\Delta ABC \sim \Delta DEF$.

SSS similarity ΔABC + ΔDEF so that

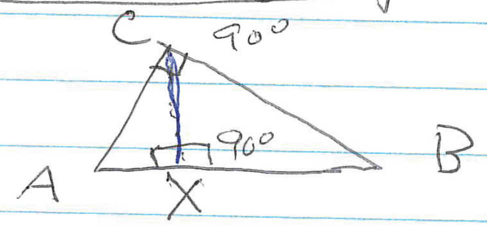
$\frac{|DE|}{|AB|} = \frac{|EF|}{|BC|} = \frac{|DF|}{|AC|} = r$, then $\Delta ABC \sim \Delta DEF$.

<u>Proof.</u> Similar to previous one.

Find ΔGHK so $\Delta ABC \sim \Delta GHK$ ratio r.

Then $|GH| = r|AB| = |DE|$, $|HK| = r|BC| = |EF|$, $|GK| = r|AC| = |DF| \Rightarrow \Delta GHK \cong \Delta DEF$ by SSS congruence, so $\Delta ABC \sim \Delta GHK \cong \Delta DEF \Rightarrow \Delta ABC \sim \Delta DEF$.

<u>Derivation of Pythagorean Thm.</u>



$X \in (AB)$ since $|\measuredangle CAB|, |\measuredangle CBA| < 90°$

$c = |AB|$, $b = |AC|$, $a = |BC|$

$h = |CX|$ $w = |AX|$, so $c - w = |BX|$.

<u>Theorem</u> (i) $\Delta AXC \sim \Delta ACB$ and

$\Delta BXC \sim$ ΔBCA

(ii) $h^2 = w(c-w)$

(iii) PYTHAGOREAN THM. $c^2 = a^2 + b^2$.

Proof (i) $|\measuredangle AXC| = |\measuredangle ACB| = 90°$ and $\measuredangle CAB = \measuredangle XAC$, so $\Delta AXC = \Delta ACB$. Switch roles of A & B to get second conclusion.

(ii) Since $\Delta AXC \sim \Delta CXB$ we have

$$\frac{u}{h} = \frac{h}{c-u} \quad \text{or} \quad h^2 = u(c-u).$$

(iii) By preceding, have

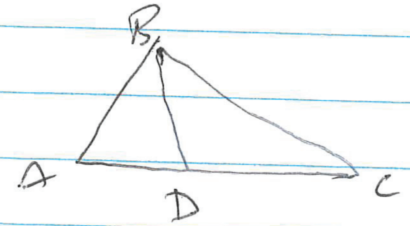
$$\frac{u}{b} = \frac{b}{c} \quad \& \quad \frac{a}{c} = \frac{c-u}{a} \quad \text{so}$$

$u = \frac{b^2}{c}$ and $c - u = \frac{a^2}{c}$. Add to get

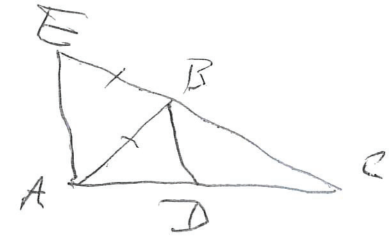
$$c = u + (c-u) = \frac{a^2+b^2}{c} \quad \text{or} \quad c^2 = a^2 + b^2.$$

Once use of similarity to discover nonobvious fact:

Given ΔABC, let $D \in (AC)$ so that $[BD$ bisects $\measuredangle ABC$. Then



$$\frac{|AB|}{|BC|} = \frac{|AD|}{|DC|}.$$



Proof.

Let E be so $E*B*C$ and $|EB| = |AB|$

($\Rightarrow |EC| = |AB| + |BC|$). By a prev. thm, BD ∥ AE and hence

$$\frac{|BC|}{|DC|} = \frac{|EB|}{|AD|} = \ldots \frac{|AB|}{|AD|} \text{ or equiv.}$$

$$\frac{|BC|}{|AB|} = \frac{|DC|}{|AD|} \text{ or } \frac{|AB|}{|BC|} = \frac{|AD|}{|DC|}.$$

2D Cartesian Coordinates

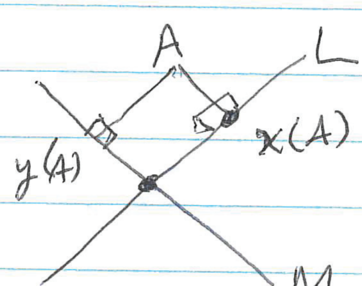
There is a 1-1 correspondence $P(\text{plane}) \to \mathbb{R}^2$

$A \to (x(A), y(A))$

so that $|AB| =$

$$\sqrt{(x(A)-x(B))^2 + (y(A)-y(B))^2}$$

Proof. Take any pair of $\perp$ lines L & M.



Get x & y from ruler fcns & $\perp$ proj.

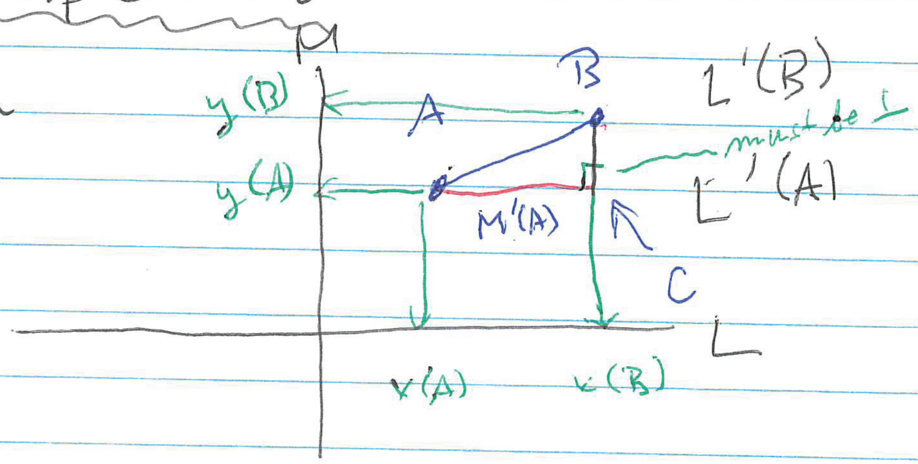
f & g ruler fcns so if $Z \in L \cap M$ then $f(Z) = g(Z) = 0$

<u>Map is 1-1</u> Say $(x(A), y(A)) = (x(B), y(B))$.
$x(A) = x(B)$ means feet of $\perp$s to L at $A + B$ are same $\Rightarrow$ same $\perp$ line M'.
Likewise $y(A) = y(B) \Rightarrow A + B$ on same line $L'(\perp M)$.
Now $L' \perp M'$; for $M' \parallel$ or $= M$ & $M \perp L \Rightarrow M'$ & $M' \perp L$, and $L' \parallel$ or $= L$ + $L \perp M' \Rightarrow L' \perp M'$.
Now $L' \cap M' =$ point, so $A = B$ in $L' \cap M'$. ∎

<u>Map is onto</u> Given $(u, v) \in \mathbb{R}^2$. Let
$U \leftrightarrow u$ on L, $V \leftrightarrow v$ on M.
$M' \perp L$ at U, $L' \perp M$ at V. Again, $L' \perp M'$ and if $A \in L' \cap M'$, then it maps to $(u, v) \in \mathbb{R}^2$.

<u>Distance interpretation</u>

Application of Pythag. Thm.



Actually 3 cases

$|AB| =$

(I+II) $x(A) = x(B)$ or $y(A) = y(B)$ — $|y(A) - y(B)|$ or

$A = C$ $B = C$ — $|x(A) - x(B)|$ resp.

(opp sides rectangle or identical segts)

(III) $x(A) \neq x(B)$ and $y(A) \neq y(B)$.

Let $C \in L'(A) \cap M'(A)$. By Pythag Thm.

$|AB| = \sqrt{|AC|^2 + |BC|^2}$ Using rectangles

see that $|AC|^2 = (x(A) - x(B))^2$ and

$|BC|^2 = (y(A) - y(C))^2$.

Angle measure is much tougher to study

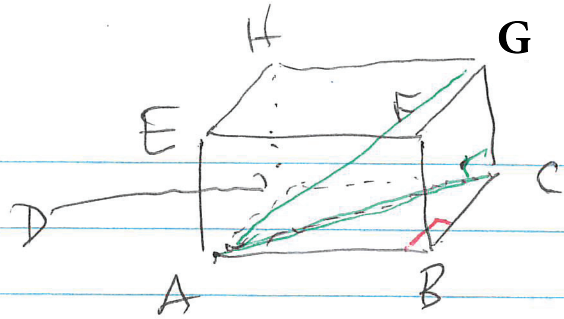
Moise sidesteps the issue. One approach — Develop plane trigonometry along lines of this course all the way to the Law of Cosines.

3D coordinates

corresponds to $(x(A), y(A), z(A))$

Need a 3D Pythagorean Thm.

Will outline.



What is $|AG|$?

Edges $\perp$ faces ∴ every line through common point on face is $\perp$ edge.

Then $|AC|^2 = |AB|^2 + |BC|^2$

But $AC \perp CG$ too, so

$|AC|^2 + |CG|^2 = |AG|^2$

Combining, $|AB|^2 + |BC|^2 + |CG|^2 = |AG|^2$.

How much choice in defining coords?

$A, B, C \in$ plane noncollinear.

Can choose so that $A \longleftrightarrow (0,0)$

$B \longleftrightarrow (u, 0)$, $u > 0$

$C \longleftrightarrow (x, y)$, $y > 0$.

Likewise in 3-dims for A, B, C, D not coplanar.