Math 133
Fall 2021

MORE SOLUTIONS FOR WEEK 04 EXERCISES

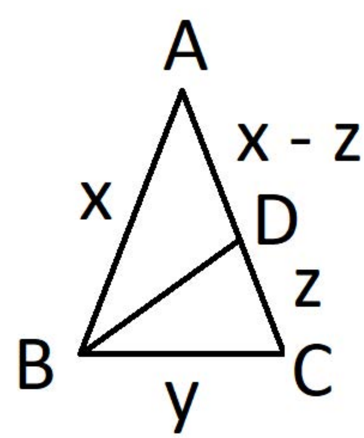
For the these exercises assume that $(\mathbf{S}; \mathcal{P}; \mathcal{L}; d; \alpha)$ or $(\mathbf{P}; \mathcal{L}; d; \alpha)$ is a system which satisfies the axioms for Euclidean geometry.

7. (a) Let L and M be perpendicular lines, and let $L \cap M = \{C\}$. Then there are two points $A, A' \in L$ so that $|AC| = |A'C| = b$, and there are also two points $B, B' \in M$ so that $|BC| = |B'C| = a$. Then $\triangle ABC$ satisfies $|BC| = a$, $|AC| = b$ and $|\angle ACB| = 90°$. This result is pretty straightforward, but it is needed for (b).■

(b) The Hinge Theorem states that if $\triangle ABC$ and $\triangle DEF$ satisfy $|AC| = |DF|$ and $|BC| = |EF|$, then $|\angle ACB| < |\angle DFE|$ if and only if $|AB| < |DE|$. We may switch the roles of the two triangles to obtain the following companion result: If $\triangle ABC$ and $\triangle DEF$ satisfy $|AC| = |DF|$ and $|BC| = |EF|$, then $|\angle ACB| > |\angle DFE|$ if and only if $|AB| > |DE|$.

By (a) we know that there is a right triangle $\triangle DEF$ such that $|AC| = |DF|$, $|BC| = |EF|$ and $|\angle DFE| = 90°$. Therefore $|DE|^2 = |DF|^2 + |EF|^2 = |AC|^2 + |BC|^2$ by the Pythagorean Theorem and the construction of $\triangle DEF$. Since two positive real numbers u and v satisfy $u < v$ if and only if $u^2 < v^2$, it follows that $|\angle ACB| < |\angle DFE| = 90°$ if and only if $|AB|^2 < |AC|^2 + |BC|^2$ and $|\angle ACB| > 90°|\angle DFE| = 90°$ if and only if $|AB|^2 > |AC|^2 + |BC|^2$.■

8. Here is a drawing:



Since $[BD$ bisects $\angle ABC$, the angle bisector theorem implies that

$$\frac{|BC|}{|AB|} = \frac{|CD|}{|AD|}$$

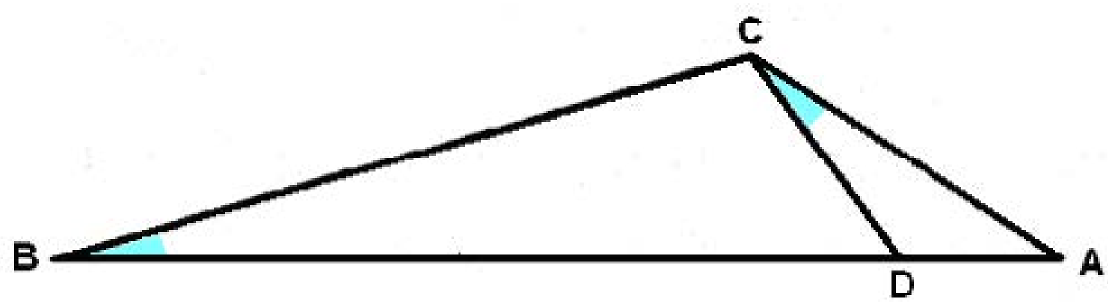
and since $A * D * C$ holds by assumption we have $|AD| = |AC| - |CD| = x - z$, so that the ratio equation becomes

$$\frac{y}{x} = \frac{z}{x - z}.$$

If we clear fractions in this equation we obtain $yx - yz = zx$, and if we solve this for z we obtain

$$z = \frac{xy}{x+y} . \blacksquare$$

9. Here is a drawing:

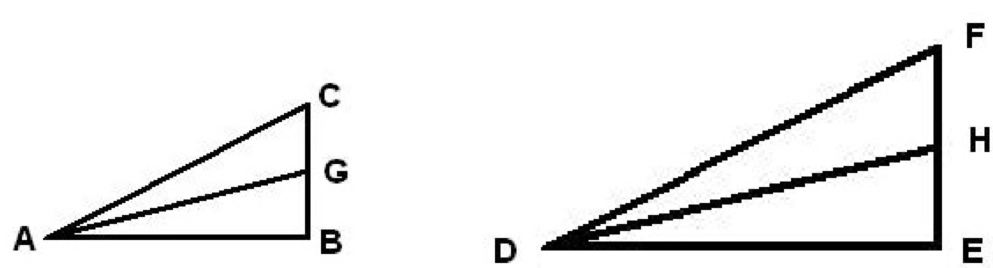


The hypothesis on angle measures and $|\angle DAC| = |\angle CAB|$ imply that $\triangle DAC \sim \triangle CAB$ by the AA Similarity Theorem. This in turn implies that

$$\frac{|AB|}{|AC|} = \frac{|AC|}{|AD|}$$

and if we multiply by both denominators we find that $|AC|^2 = |AB| \cdot |AD|$, which is what we wanted to prove.$\blacksquare$

10. Here is a drawing:



Let r be the ratio of similitude, so that

$$\frac{|DE|}{|AB|} = \frac{|EF|}{|BC|} = \frac{|DF|)}{|AC|} = r$$

and note that $\angle GBA = \angle CBA$ and $\angle HED = \angle FED$, so that $|\angle ABG| = |\angle DEH|$. Since G and H are midpoints of $[BC]$ and $[EF]$, we have $|BG| = \frac{1}{2}\,|BC|$ and $|EH| = \frac{1}{2}\,|EF|$, so that

$$\frac{|EH|}{|BG|} \quad = \quad \frac{|EF|}{|BC|} \quad = \quad r \; .$$

Combining these, we can apply the SAS similarity theorem to conclude that $\triangle ABG \sim \triangle DEH$ with ratio of similitude r.■

11. We know that $\triangle ABC$ is isosceles, and by permuting the vertices if necessary we may assume that $|AB| = |AC|$ and $\triangle ABC \sim \triangle DEF$. Therefore we have

$$\frac{|AB|}{|DE|} \quad = \quad \frac{|AC|}{|DF|} \qquad \text{and thus also} \qquad \frac{|AB|}{|AC|} \quad = \quad \frac{|DE|}{|DF|} \; .$$

Since $\triangle ABC$ is isosceles the left hand side of this equation is equal to 1, which means that $|DE| = |DF|$ and hence $\triangle DEF$ is also isosceles.■

12. By the main result on setting up coordinates, we can construct a coordinate system for an arbitrary pair of perpendicular lines L and M and ruler functions $f : L \to \mathbb{R}$ and $g : M \to \mathbb{R}$; in this construction the common point of L and M corresponds to $(0, 0)$. Starting with noncollinear points A, B, C in the given plane, take $L = AB$ and M to be the perpendicular to L at A, and let $f_0 : L \to \mathbb{R}$ and $g_0 : M \to \mathbb{R}$ be ruler functions for L and M respectively such that $f_0(A) = 0 = g_0(A)$.

If D is the foot of the perpendicular L' to M through C, we claim that $D \notin L$. First note that $C \in L'$ but $C \notin L$ by construction. Now $L \neq L'$, and since both are perpendicular to M it follows that L' is parallel to L hence $D \notin L$, so that $D \neq A$. Therefore $D \notin L$ as claimed. Multiplying each of f_0 and g_0 by -1 if necessary, we obtain ruler functions f and g such that $f(B) > 0$ and $g(C) = g(D) > 0$. It follows that A corresponds to $(0, 0)$, B corresponds to $(u, 0)$ for some $u > 0$, and C corresponds to (x, y) where $y > 0$.■