Using coordinates I - Circles

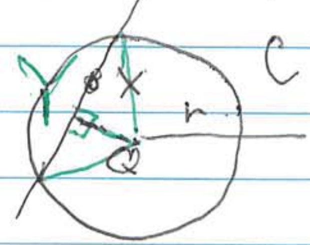
Moise, Ch. 16

Circle with center Q, radius r = all points X so $|QX| = r$. Interior, exterior given by $|QX| < r$, $|QX| > r$.

Emphasis on two results used in constructions with (unmarked) straightedge and compass.

Line - Circle Theorem C = circle, Q center & r radius, L some line. If $X \in L \cap \mathrm{Int}(C)$, then $L \cap C$ = 2 points.

Proof.



Take $\perp$ to L through X.

$X = Q$ $f: L \to \mathbb{R}$ ruler, $f(Q) = 0$.

Choose A + B so $f(A) = r$, $f(B) = -r$.

$X \neq Q$ $|QX| < r$ given. Drop $\perp$ to L from X, foot = Y. Then $|QY| \leq x < r$ (maybe $X = Y$, but doesn't matter).

$f: L \to \mathbb{R}$ ruler so $f(Y) = 0$.

Take A, B so $f(A \text{ or } B) = \pm\sqrt{r^2 - |QY|^2}$

positive!

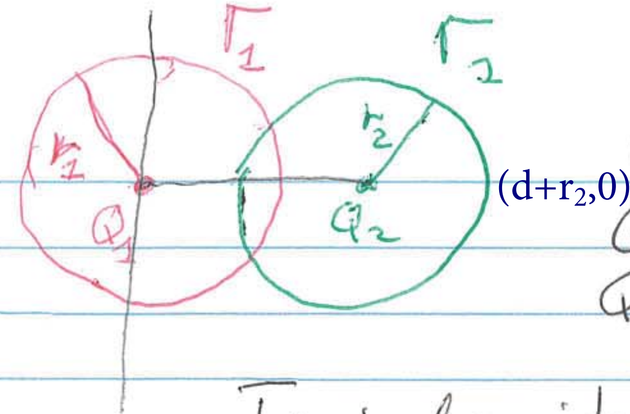
By Pythagorean Thm., $A, B \in C$.

This is relevant to constructions with straightedge and compass, guaranteeing the existence of meeting points.

Another result on this topic.

<u>Two Circle Theorem</u> Suppose given circles Γ_1 & Γ_2 such that Γ_2 contains a point inside Γ_1 and a point outside Γ_2. Then $\Gamma_1 \cap \Gamma_2$ is 2 points, one on each side of the line joining their centers.

<u>Note</u> This property is tacitly assumed in the very first proposition of Euclid's <u>Elements</u> which constructs equilateral triangles.



Chose coordinates so
$Q_1 \leftrightarrow (0,0)$
$Q_2 \leftrightarrow (d,0)$ with $d>0$.

Typical point on Γ_2 is $(x-d,y)$ where $(x-d)^2+y^2=r_2^2$. Want to know when point inside, on, outside Γ_1. Algebraically want to know when $x^2+y^2 \left\{\begin{matrix} < \\ = \\ > \end{matrix}\right\} r_1^2$. By preceding $x^2+y^2 = r_2^2+2xd-d^2 = h(x)$

Minimum value occurs for minimum x, which is $d-r_2$, maximum when x is $d+r_2$.

We are given one pt inside Γ_1, another outside

$$r_2^2-2r_2d+d^2 \le h(u) < r_1^2 < h(v) < r_2^2+2r_2d-d^2$$

which means $|r_2-d| < r_1 < r_1+d$.

Now $h(x)=r_1^2$ translates to

$$r_1^2 = r_2^2+2xd-d^2$$ and hence

$$x = r_1^2+d^2-r_2^2/2d.$$

For the conclusion to hold, necc. & suff that

$|x| < r_1$ and hence

$$-2dr_1 < r_1^2 + d^2 - r_2^2 < 2dr_1.$$

This is equiv to

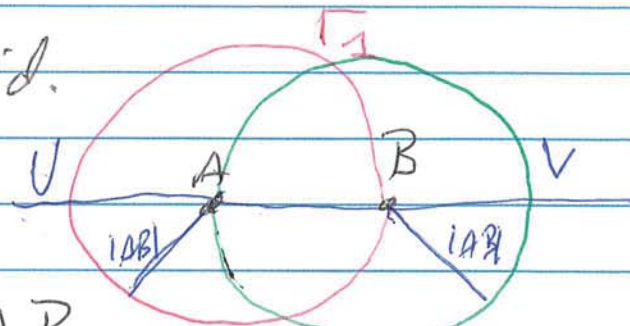
$$-(r_1+d)^2 < -r_2^2 < -(r_1-d)^2$$

$$(r_1-d)^2 < r_2^2 < (r_1+d)^2$$

We already know this $\rightarrow$ $|r_1 - d| < r_2 < r_1 + d.$

Hence we have a unique solution for x, and if $y = \pm\sqrt{r_1^2 - x^2}$, then $(x, \pm y)$ are the two points of $\Gamma_1 \cap \Gamma_2$.

Use in Euclid.



$|AB|$ = common radius

Choose $U, V \in AB$

so $U * A * B$, $A * B * V$ $|UB| = |AV| = 2|AB|$.

This yields X and Y so ΔABX + ΔABY equilat.

Converse to triangle inequality

$0 < a \le b \le c$.

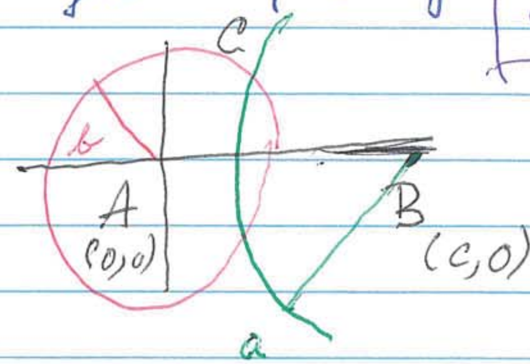
Then there is ΔABC with $|AB| = c$, $|AC| = b$

$|BC| = a \iff c < a + b$.

We automatically have
$b \le c < a+c$
$a \le c < b+c$

($\Rightarrow$) is the triangle inequality.

Proof of ($\Leftarrow$)



Apply two circle thm.

Γ_1 centered at A

Γ_2 at B

Γ_2 has point inside Γ_1 $(c-a, 0)$

since $c - a < b$

Also outside Γ_1 $(c+a, 0)$ since

$a + c \ge a + b > b$.