Math 133
Fall 2021

SOLUTIONS FOR WEEK 05 EXERCISES

Assume that $(\mathbf{S}; \mathcal{P}; \mathcal{L}; d; \alpha)$ or $(\mathbf{P}; \mathcal{L}; d; \alpha)$ is a system which satisfies the axioms for Euclidean geometry.

1. Let Γ be a circle with center Q. We shall prove the theorems in order.

Theorem 4. *If AB is a chord in Γ and $C \neq Q$ is the midpoint of $[AB]$, then $QC \perp AB$.* — This follows because both Q and C are equidistant from A and B, so that they lie on the perpendicular bisector of $[AB]$. Since $Q \neq C$ this means that $QC \perp AB$. (We have added the condition $Q \neq C$ because the result is either trivial or not meaningful if $Q = C$.)■

Theorems 6 – 7. *If $[AB]$ and $[CD]$ are chords of Γ which are equidistant from the center, then $|AB| = |CD|$. Conversely, if $|AB| = |CD|$ then $[AB]$ and $[CD]$ are chords which are equidistant from the center of Γ.* — As in the proof of Theorem 4, one must distinguish the cases when Q lies on the center of a chord and when it does not. The following observation implies that either both chords are diameters or neither chord is a diameter.

LEMMA. *In a circle of radius r, the length of a chord in the circle is $2r$ if and only if the chord is a diameter, and the length of the chord is strictly less than $2r$ if the chord is not a diameter.*

Proof of Lemma. If a chord is a diameter, then the center of the circle is its midpoint, and it follows thaat the length of the chord must be $2r$. Assume now that the chord $[AB]$ is not a diameter. Let C be the chord's midpoint. We know that $QC \perp AB$ by Theorem 4, and therefore we have a right triangle $\triangle QAC$ whose hypotenuse is QA. In a right triangle the hypotenuse is the longest side (either the Pythagorean Theorem or the result on larger angles versus longer sides), and therefore $r = |QA| > |QC|$. Therefore we must have $|AB| = 2|AC| < 2|QA| = 2r$.■

Back to the proofs of the theorems. The distance from Q to the chord is zero if and only if the chord $|AB|$ is a diameter, and we have shown that the latter is equivalent to $|AB| = 2r$. Therefore the distance is zero if and only if the chord length is $2r$. It remains to consider the case where neither chord is a diameter, so their distances from Q are positive and their lengths are strictly less than $2r$.

Let E and F denote the midpoints of $[AB]$ and $[CD]$ respectively, and assume first that the chords are equidistant from the center Q and this distance is positive. Then the hypotenuse-side congruence theorem for right triangles implies $\triangle QEA \cong \triangle QFC$, so that $|AE| = |CF|$. This yields the identities

$$|AB| \;=\; 2\,|AE| \;=\; 2\,|CF| \;=\; |CD|$$

and hence the chords have equal length.

Conversely, assume the chords have equal length less than $2r$, and let E and F be as in the preceding paragraph. Then we have $|AE| = \frac{1}{2}\,|AC| = \frac{1}{2}\,|BD| = |BF|$, so in this case we also have $\triangle QEA \cong \triangle QFC$ by the hypotenuse-side congruence theorem for right triangles. Therefore $|QE| = |QF|$, which is what we wanted to prove.■

2. As in the previous exercise, one must distinguish the cases when Q lies on the center of a chord and when it does not.

Assume first that $Q \in AB$, and choose a ruler function f such that $f(Q) = 0$. If $r > 0$ is the radius of Γ, then $f(A) = r$ and $f(B) = -r$ or vice versa. It follows that a point $X \in AB$ lies in the interior of Γ if $|f(X)| < r$ and X lies in the exterior of Γ if $|f(X)| > r$. The first case is equivalent to the betweenness relation $A * C * B$ and the second case is equivalent to $A * B * C$ or $C * A * B$.

Assume now that $Q \notin AB$, so that it is meaningful to discuss $\triangle QAB$. If $A * C * B$, then $|\angle QAB| = |\angle QBA| < |\angle QCA|$ by the Exterior Angle Theorem, and therefore $|QC| < |QA| = r$ because the larger of two angles in a triangle is opposite the longer side. If $A * B * C$, then $|\angle QAB| = |\angle QBA| < |\angle QCA|$ by the Exterior Angle Theorem, and therefore $|QC| > |QA| = r$ because the larger of two angles in a triangle is opposite the longer side. If $B * A * C$ and we switch the roles of A and B in the preceding argument, we see that $|QC| > |QB| = r$.■

3. The betweenness conditions in the problem imply that A and D lie on opposite sides of BC, and $|\angle AXB| = |\angle DXC|$ by the Vertical Angle Theorem. Therefore we also have $\triangle AXB \cong \triangle DXC$, so that $|\angle XAB = \angle DAB| = |\angle XDC = \angle ADC|$. If we view BC as a transveral to the lines AB and CD, this means that we have a pair of alternate interior angles satisfying $|\angle DAC| = |\angle ADC|$, and by the neutral half of the theorem on transversals we know that AB is parallel to CD.■

4. (a) Choose a coordinate system such that the line is the x-axis and the circle is defined by the equation $x^2 + (y - h)^2 = r^2$ where r is the radius and $h > 0$; these mean that the center of the circle is $(0, h)$. If the intersection of the circle with the line is at least two points we must hae $r > h$.

Assume now that the circle and the line meet at points $(a, 0)$, $(b, 0)$ and $(c, 0)$. Then we have $a^2 + h^2 = b^2 + h^2 = c^2 + h^2 = r^2$, so that

$$a^2 \quad = \quad b^2 \quad = \quad c^2 \quad = \quad r^2 - h^2$$

so that each of a, b, c is equal to $\pm\sqrt{r^2 - h^2}$. Since there are only two possible solutions for a, b, c it follows that at least two of these numbers must be equal, and hence there can only be two points which lie on both the line and the circle.■

(b) Choose coordinates so that the centers of the circles are $(0, 0)$ and $(a, 0)$, so the common points satisfy the equations $x^2 + y^2 = p^2$ and $(x - a)^2 + y^2 = q^2$. If we subtact the first equation from the second we obtain the linear equation $2ax + a^2 = q^2 - p^2$ which yields

$$x \quad = \quad \frac{q^2 - p^2 - a^2}{2a} .$$

The corresponding values of y can be retrieved from the formula $y = \pm\sqrt{p^2 - x^2}$. Hence there are at most two solutions to the system of equations, so that there are also at most two points where the circles intersect.■

5. By the Line-Circle Theorem, the line AX meets the circle Γ in two points, say B and C, and Exercise 2 implies that $B * A * C$ is true. This line is a union of the rays $[AB$ and $[AC$, and one of the following must be true:

(1) Both $B \in [AX$ and $C \notin [AX$.

(2) Both $B \notin [AX$ and $C \in [AX$.

In each case exactly one of the points $\{B, C\}$ lies in $[AX \cap \Gamma$; since no other points lie in $AX \cap \Gamma$, it follows that $[AX \cap \Gamma$ consists of a single point.■