Math 133
Fall 2021

MORE EXERCISES FOR WEEK 06

For these exercises assume that the coordinate plane satisfies the axioms for Euclidean plane geometry, with distances and lines as described in Chapter 17 of Moise. Most of what we need about rigid motions and similarities appears in `lecture12.pdf`.

Definition. If F_1 and F_2 are subsets of a neutral geometry, then a 1–1 onto mapping $h : F_1 \to F_2$ is called an *isometry* (or a *congruence*) if for all $X, Y \in F_1$ we have $h(X)h(Y)| = |XY|$. — All of the exercises below are valid in a neutral geometry.

0. Explain why the argument in `lecture12.pdf` generalizes to show that an isometry preserves collinear points and betweenness of three collinear points. Also, explain why (i) The inverse map h^{-1} is an isometry and the composite of two isometries is an isometry.

1. If $h : F_1 \to F_2$ is an isometry and $E \subset F_1$ is a maximal collinear subset of F_1, prove that $h[E]$ is a maximal collinear subset of F_2.

2. (a) Let $h : \angle ABC \to \angle DEF$ be an isometry. Explain why h maps $[BA$ to either $[ED$ or $[EF$ and it maps $[BC$ to the other of $[ED$ and $[EF$. Why does it follow that $h(B) = E$? Give examples to show that $h(A)$ and $h(C)$ are not necessarily equal to E and F.

(b) Assume the same setting as in (a). Prove that $|\angle ABC| = |\angle DEF|$. [*Hint:* Use the SSS Congruence Theorem.]

3. (a) Let $\diamondsuit ABCD$ be a **square**; in other words, $\diamondsuit ABCD$ is a convex quadrilateral such that all four sides (edges) have the same length and all four vertex angles are right angles. Suppose also that $h : \diamondsuit ABCD \to \diamondsuit ABCD$ is an isometry. Explain why h maps the set of vertices $\{A, B, C, D\}$ into itself, and if $h(A) = A$, then either $h(B) = D$ and $h(D) = C$ or else $h(B) = D$ and $h(D) = B$, so that $h(C) = C$ holds in all cases. Similarly, show that there are two possibilites if $h(A) = B$, $h(A) = C$ or $h(A) = D$.

(b) If in (a) we know that $h(A) = A$, $h(B) = B$, $h(C) = C$ and $h(D) = D$, show that h is the identity map.

4. (a) A subset F of a neutral geometry has a **proper diameter** M if $|XY| \leq M$ for all $X, Y \in F$ and there are points $X_0, Y_0 \in F$ such that $|X_0Y_0| = M$. If $h : F_1 \to F_2$ is an isometry and F_1 has a proper diameter equal to M, prove that F_2 also has a proper diameter equal to M.

(b) Suppose that $f : [a, b] \to [c, d]$ is an isometry. Prove that $b - a = d - c$.

(c) Prove that there is no isometry from the closed interval $[a, b]$ to the open interval (a, b).

(d) Let Γ_1 and Γ_2 be circles whose radii are equal to a and b respectively, and suppose that $h : \Gamma_1 \to \Gamma_2$ is an isometry. Prove that $a = b$.