Math 133
Spring 2022
L12B

VERIFICATION OF PLANAR INCIDENCE AXIOMS

First, a reminder of what we have done before and are now trying to do. Using the Pythagorean Theorem we have shown that a system which satisfies the axioms for a Euclidean plane must be in a 1–1 date-preserving correspondence with $\mathbb{R}^2$. Howeer, this does **NOT** guarantee that all the theorems in Euclidean geometry are automatically true in the coordinate model. In order to ensure the latter is true, one needs to verify specifically that the coordinate data satisfy all the axioms for the Euclidean plane. This is tedious and sometimes too complicated for a course at this level, so we shall simply illustrate how one verifies the planar Axioms of Incidence along the lines of Moise, Chapter 26 (especially Sections 26.2 and 26.3).

(I–1): *There is a unique line joining a given pair of points.*

The existnce of a line is shown in Theorem 26.3.1 of Moise. Here is how we prove uniqueness: Suppose that the points in question are $(a_1, a_2) \neq (b_1, b_2)$. Subcase 1. Suppose that $a_1 = b_1$, so that $a_2 \neq b_2$. The only vertical line containing both points is the line $x = a_1 (= b_1)$; we claim there are no nonvertical lines containing both points. If $y = mx + c$ isi the equation of a nonvertical line and $a_1 = b_1$ then we have $b_2 = mb_1 + c = ma_1 + c = a_2$; this contradicts $a_2 \neq b_2$, and therefore one cannot have a nonvertical line containing both (a_1, a_2) and (b_1, b_2). Subcase 2. Suppose that $a_1 \neq b_1$; then there is no vertical line $x = K$ containing both points. As noted in the first sentence of the proof, there is at least one line which contains both points, and we know it must be nonvertical; likewise, if there is a second line containing the points, this line must also be nonvertical. Suppose now that the lines $y = mx + c$ and $y = Mx + C$ contain (a_1, a_2) and (b_1, b_2). We can now apply Theorem 26.3.2 of Moise to conclude that the two lines are the same.■

(I–2): *Every line contains at least two points.*

This is even easier to check. A vertical line of the form $x = a$ always contains the points $(a, 0)$ and $(a, 1)$. A nonvertical line with equation $y = mx + b$ always contains the points $(0, b)$ and $(1, m + b)$.■