Axiomatic Systems

Neutral geometry $(\mathcal{P}, \mathcal{L}, d, \alpha)$ or $(\mathcal{S}, \mathcal{L}, \mathcal{P}, d, \alpha)$ are the data. They satisfy

Incidence Axioms
Distance and Ruler Axioms
Plane Separation Axiom
Angle Measurement Axioms
Triangle Congruence Axioms

Euclidean geometry also satisfies

Euclidean Parallel Postulate.

Long standing question:
Does neutral $\Rightarrow$ Euclidean?

Answer shown to be NO in 19th century.

What does this mean?

Need more insight into axiomatic math systems, given by

DATA Usually some set and subsets, functions ...

a.k.a. Postulates Assumptions

AXIOMS Rules which the data satisfy.

MINIMAL PROPERTY Must be (relatively) logically consistent.

HOW TO SHOW Build example within set theory satisfying the given conditions. MODEL

For example, start with counting #s $\mathbb{N}$, show existence of negatives consistent by constructing $\mathbb{Z}$ from them.

WHY RELATIVELY? Work of K. Gödel about 1930 shows we can never be sure that the standard axioms for $\mathbb{N}$ (due to G. Peano) are consistent. Problem arises from infinite nature of $\mathbb{N}$. (Thus far, no examples beyond "artificial" ones.)

REDUNDANCY QUESTION Given a system with data D and axioms A, let S be a meaningful statement about the system. What does it mean to say that S cannot be proved as a theorem in the system?

ANSWER Construct a model of the system in which S is false.

Example from incidence geometry

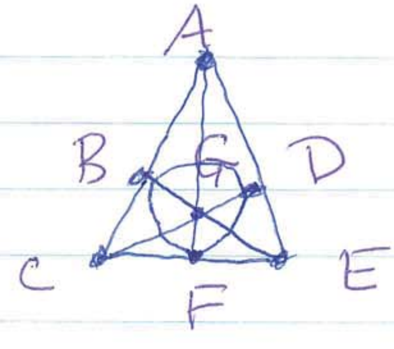
Consider the statement,

Given a line L and point $X \notin L$, there is a line M so that $X \in M$, $M \subseteq$ plane of L and x, and $L \cap M = \phi$.

Proof that ~~result~~ statement is a not a consequence of the axioms Construct an incidence plane in which every pair of lines has a common point.

Look at the 7 point plane from early in the course

Notice there are also 7 lines



Verifying the axioms is tedious but straightforward.

For example, $\{A, B\} \subseteq \{A, B, C\}$ but not $\{A, D, E\}$, $\{B, D, F\}$, $\{A, G, F\}$, $\{B, G, E\}$, $\{C, G, D\}$ or $\{C, F, E\}$.

There are $\binom{7}{2}-1$ other point pairs that must be considered. Similarly one must look at $\binom{7}{2}$ intersections of two lines. $=20$

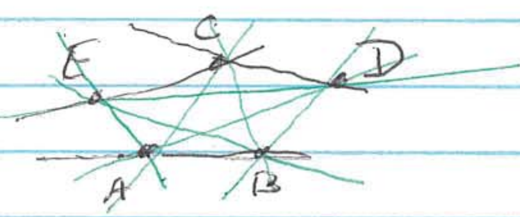
STONE AGE METHOD! There are far more efficient ways to verify this example!!

Note also Existence of at **most** one parallel does not follow.

$\mathcal{P}$ = set with ≥ 5 elements

$\mathcal{L}$ = two point subsets

Both CD and CE are parallel to AB



5 points

$10 = \binom{5}{2}$ lines

Long vs. short lists of axioms/data

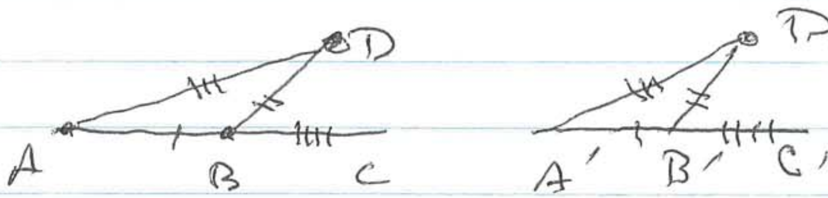
Advantages LONG Faster development of material. (Examples: Angular measure, area axioms in 2D, volume axioms in 3D).

SHORT Isolates key ideas more clearly, easier to test for logical consistency.

Disadvantages LONG Much redundancy, risk of information overload, contrary in spirit to Euclid's Elements.

SHORT: Proofs that some axioms in long lists can be extremely long and nonelementary. (Example: Verification of 3D volume axioms requires concepts from integral calculus).

Angle measurement turns out to be redundant if we assume incidence, distance/ruler, plane/space separation plus the following



In this situation, $|CD| = |C'D'|$.

REQUIRES LONG DIGRESSIONS!!

Categorical vs. noncategorical axiom systems

Categorical Given two systems satisfying the data and axiom stipulations, each is structurally equivalent to the other.

Example Euclidean planes or 3-spaces
1-1 correspondence, preserving lines, nonlines (+ planes, nonplanes), distance, angle measure.

However, sometimes we want noncategorical data/axiom combinations.

Examples Systems with $+, -, \times$ (maybe not $\div$) and distributive laws

$a(b+c) = ab + ac$, $(a+b)c = ac + bc$

number systems, polynomials, sq. matrices.*

*(where ab & ba need not be equal).

Yields results valid in a broad range of examples.

Last but not least Euclidean geometry is consistent (Moise, Ch. 26).

One simple step (I1) Unique line in plane joining two points $A = (a_1, a_2)$, $B = (b_1, b_2)$.

Define L by

$$\text{“} \frac{y - a_2}{x - a_1} = \frac{b_2 - a_2}{b_1 - a_1} \text{”}$$ More correctly,

$$(y - a_2)(b_1 - a_1) = (x - a_1)(b_2 - a_2).$$

Check this is the unique line containing A & B.