Math 133
Fall 2021

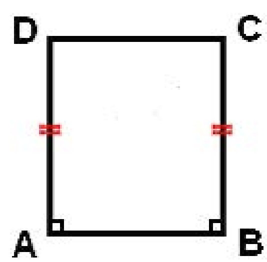
EXERCISES FOR WEEK 8

For these exercises assume that all points lie in a plane which satisfies the axioms for neutral geometry.

1. Prove the following consequence of the Archimedean Law that was stated in the notes: *If h and k are positive real numbers, then there is a positive integer n such that $h/2^n < k$.* [*Hint:* Why is there a positive integer n such that $1/n < k/h$? Use this and the inequality $n < 2^n$.]

2. Suppose that p and q are arbitrary positive real numbers. Prove that there is a Saccheri quadrilateral $\diamondsuit ABCD$ with base AB such that $|AD| = |BC| = p$ and $|AB| = q$.

Standing hypotheses: In Exercises 2–5 below, points A, B, C, D in a neutral plane form the vertices of a Saccheri quadrilateral such that AB is perpendicular to AD and BC, and $|AD| = |BC|$. The segment $[AB]$ is called the *base*, the segment $[CD]$ is called the *summit*, and $[AD]$ and $[BC]$ are called the lateral sides. The vertex angles at C and D are called the *summit angles*.



3. Prove that the summit angles at C and D have equal measures. [*Hint:* Why do the diagonals have equal length? Use this fact to show that $\triangle BDC \cong \triangle ACD$.]

4. Let X and Y be the midpoints of $[AB]$ and $[CD]$ respectively. Prove that the line XY is perpendicular to both $[AB]$ and $[CD]$. [*Hint:* Why does it suffice to prove that Y is equidistant from A and B and X is equidistant from C and D?]

5. In the given setting, prove that if we $|AB| = |CD|$ then the Saccheri quadrilateral $\diamondsuit DABC$ is a rectangle.

6. Suppose we are given Saccheri quadrilaterals $\diamondsuit DABC$ and $\diamondsuit HEFG$ with right angles at$A$, B and$E$, F such that the lengths of the bases and lateral sides in $\diamondsuit DABC$ and $\diamondsuit HEFG$ are equal. Prove that the lengths of the summits and the measures of the summit angles in $\diamondsuit DABC$ and $\diamondsuit HEFG$ are equal.

7. Suppose are given Lambert quadrilaterals $\diamondsuit ABCD$ and $\diamondsuit EFGH$ with right angles at A, B, C and E, F, G such that$|AB| = |EF|$ or $|BC| = |FG|$. Prove that$|CD| = |GH|$, $|AD| = |EH|$, and$|\angle CDA| = |\angle GHE|$.

8. Suppose that the points A, B, C, D form the vertices of a Lambert quadrilateral with right angles at A, B, C. Prove that $|AD| \leq |BC|$ and $|AB| \leq |CD|$. [*Hint:* Start by explaining why it suffices to prove the first of these. Show that there is a Saccheri quadrilateral $\Diamond DAEF$ such that B and C are the midpoints of the base $[AE]$ and the summit $[FD]$ respectively. Apply Exercise 4.]

9. In the setting of the preceding exercise, prove that if we have $|AB| = |CD|$ or $|AD| = |BC|$, then the Lambert quadrilateral $\Diamond ABCD$ is a rectangle.

10. Let p and q be arbitrary positive real numbers. Prove that there is a Lambert quadrilateral $\Diamond ABCD$ with right angles at A, B and C such that $|AD| = p$ and $|AB| = q$. [*Hint:* One can view a Lambert quadrilateral as half of a Saccheri quadrilateral.]