Math 133
Fall 2021

MORE EXERCISES FOR WEEK 8

For these exercises the geometrical settings vary, so make sure you are working in the right type of system.

11. Suppose that we are given $\triangle ABC$ and $\triangle A'B'C'$ in a neutral plane $\mathcal{P}$ such that $\triangle ABC \cong \triangle A'B'C'$. Let D, E, F be the respective midpoints of $[BC]$, $[AC]$ and $[AB]$, and let D', E', F' be the respective midpoints of $[B'C']$, $[A'C']$ and $[A'B']$.

(a) Prove that $\triangle AEF \cong \triangle A'E'F'$, and explain why we must also have $\triangle BDF \cong \triangle B'D'F'$ and $\triangle CDE \cong \triangle C'D'E'$.

(b) Prove that $\triangle DEF \cong \triangle D'E'F'$.

(c) In Euclidean geometry one can improve the preceding results to say that all eight triangles in (a) and (b) are congruent to each other (with the vertices suitably ordered). Write out the conclusion explicitly, and explain why it is true.

12. (a) Given a Saccheri quadrilateral in a hyperbolic plane $\mathcal{P}$, explain why the summit is always longer than the base. If $\diamondsuit ABCD$ is a Lambert quadrilateral in a hyperbolic plane $\mathcal{P}$ with right angles at A, B, and C, what can one say about the lengths of the pairs of opposite sides $\{[AB], [CD]\}$ and $\{[BC], [AD]\}$? Give reasons for your answer.

(b) Given a Saccheri quadrilateral in a hyperbolic plane $\mathcal{P}$, show that the line segment joining the midpoints of the summit and base is shorter than the lengths of the lateral sides.

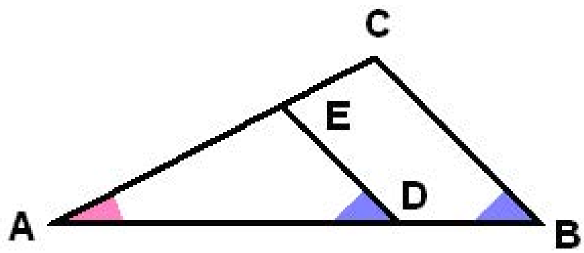
(c) In Euclidean geometry all Saccheri and Lambert quadrilaterals are rectangles. Explain why no quadrilateral in a hyperbolic plane can be both a Saccheri quadrilateral and a Lambert quadrilateral.

13. Suppose that h is an arbitrary positive real number and $\mathcal{P}$ is a hyperbolic plane. Prove that there is a triangle in $\mathcal{P}$ whose angle defect is less than h. [*Hint:* Let $\triangle ABC$ have defect $\delta > 0$. If one splits it into two triangles, why will at least one of them have defect at most $\delta/2$? Show that if one iterates this enough times, one obtains the desired triangle.]

14. (a) Suppose we are given an isosceles triangle $\triangle ABC$ in a hyperbolic plane $\mathcal{P}$ with $|AB| = |AC|$, and let D and E be points on (AB) and (AC) respectively such that $|AD| = |AE|$. Prove that $|\angle ABC| < |\angle ADE|$. [*Hint:* Compare the angle defects of the two isosceles triangles in the problem.]

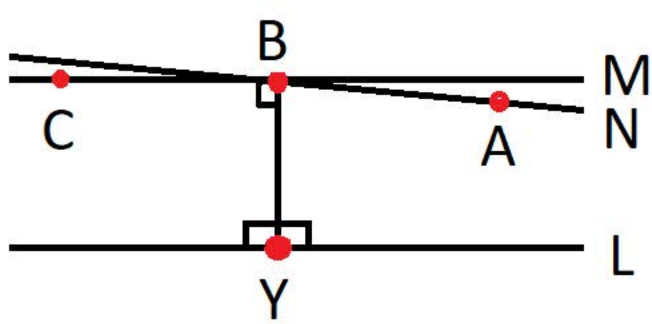
(b) Suppose we are given an equilateral triangle$\triangle ABC$ in a hyperbolic plane $\mathcal{P}$, and let D, E and F be the midpoints of $[BC]$, $[AC]$ and $[AB]$. Prove that $\triangle DEF$ is also an equilateral triangle and that$|\angle ABC| < |\angle DEF|$. Using the conclusion of (a), explain why $\triangle AEF$ is not an equilateral triangle.

15. Prove the following result, which shows that the "The third angles are equal" property in Euclidean geometry fails completely in hyperbolic geometry: Given a triangle $\triangle ABC$ in a hyperbolic plane $\mathcal{P}$ and a point D on (AB), then there exists a point $E \in (AC)$ such that $|\angle ABC| = |\angle ADE|$ but also $|\angle ACB| < |\angle AED|$.



16. Suppose we are given a Saccheri quadrilateral $\diamondsuit ABCD$ in a hyperbolic plane $\mathcal{P}$ with base $[AB]$, and assume that the lengths of the base and lateral sides are equal. Does the ray $[AC$ bisect $|\angle DAB|$? Give reasons for your answer.

17. (a) Suppose we are given a hyperbolic plane $\mathcal{P}$. Prove that there is a line L and an angle $\angle ABC$ in $\mathcal{P}$ such that L is contained in the interior of $\angle ABC$.



[*Hint:* Let B be a point not on L such that there are at least two parallel lines to L through B. If Y is the foot of the perpendicular from B to L and M is a line through B which is perpendicular to YB, then M is parallel to L, and there is also a second line N which is parallel to L. Explain why there is a ray $[BA$ on N such that A lies on the same side of M as Y, and explain why there is also a ray $[BC$ on M such that C and A lie on opposite sides of BY. Why is L disjoint from $\angle ABC$, why does the point Y on L lie in the interior of this angle, and why does this imply that the entire line is contained in the interior of $\angle ABC$?]

(b) Suppose we are now given a Euclidean plane $\mathcal{P}$. Let L be a line in $\mathcal{P}$, and let $\angle ABC$ be an angle in $\mathcal{P}$. Prove that L is not contained in the interior of $\angle ABC$. [*Hints:* Why does it suffice to prove the result when L contains at least one point in the interior of the angle? And why does at least one of the lines AB and BC contain a point of L?]