Before Gauss, some mathematicians (for example, Klügel) had speculated that a proof of the Fifth Postulate might be out of reach. However, Gauss (and to a lesser extent a contemporaries like Schweikart and Taurinus) took things an important step further, concluding that the negation of the Fifth Postulate yields a geometrical system which is very different from Euclidean geometry in some respects but has ***exactly the same degree of logical validity*** (compare also the passage from the letter to Olbers at the beginning of this unit). Working independently of Gauss, J. Bolyai (1802 – 1860) and N. I. Lobachevsky (1792 – 1856) reached the same conclusions as Gauss (each one independently of the other), which Bolyai summarized in a frequently repeated quotation:

Out of nothing I have created a strange new universe.

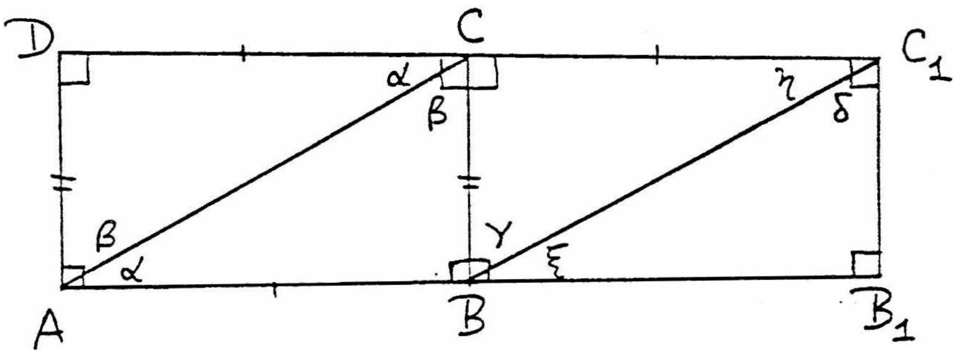
Both Bolyai and Lobachevsky took everything one important step further than Gauss by publishing their conclusions, and for this reason they share credit for the first published recognition of hyperbolic geometry as a mathematically legitimate subject.

Details for a result from Lecture 15 (note fixed typos)

Appendix to Section 3: Proof of Theorem 8

The major steps in the argument will be presented as lemmas.

Lemma 8A. (Splicing Property). *Suppose that* $\square$**ABCD** *is a rectangle, and let* $\mathbf{C_1} \in$ **(DC** *be a polnt such rhat* $|\mathbf{DC_1}| = 2|\mathbf{DC}|$. *Let* $\mathbf{B_1}$ *be the foot of the perpendicular from* $\mathbf{C_1}$ *to* **AB.** Then $|\angle \mathbf{DC_1B}| = 90°$ *and the points* **A**, $\mathbf{B_1}$, $\mathbf{C_1}$ *and* **D** *(**in that order**) are the vertices of a rectangle.*



Proof. First of all, the lines **AD, BC,** and $\mathbf{B_1C}$, are all parallel to each other because every two of them have a common perpendicular (namely, **AB**). Therefore **AD** and $\mathbf{B_1C_1}$ are contained in the **D** – and $\mathbf{C_1}$ – sides of **BC** respectively. But $|\mathbf{DC_1}| = 2|\mathbf{DC}|$ and $\mathbf{C_1} \in$ **(DC** imply $\mathbf{D*C*C_1}$ is true. This in turn implies that **D** and $\mathbf{C_1}$ are on opposite sides of **BC.** Since **B** is the common point of the lines $\mathbf{AB_1}$ and **BC,** it follows that $\mathbf{A*B*B_1}$ is true.

Since **AD** and $\mathbf{B_1C_1}$ are parallel (they have a common perpendicular), the points $\mathbf{B_1}$ and $\mathbf{C_1}$ lie on the same side of **AD.** Hence **A**, $\mathbf{B_1}$, $\mathbf{C_1}$, and **D** (in that order) form the vertices of a convex quadrilateral. Likewise **B**, $\mathbf{B_1}$, $\mathbf{C_1}$, and **C** form the vertices of a convex quadrilateral. By construction, **S.A.S.** applies to show $\triangle\mathbf{ADC} \cong \triangle\mathbf{BCC_1}$. It follows that $|AC| = |BC_1|$, $\gamma = |\angle CBC_1| = |\angle DAC| = \alpha$, and $\eta = |\angle BC_1C| = |\angle ACD|$.

On the other hand, if $\xi = |\angle C_1C B_1|$ then

$$\alpha + \beta = 90° = \gamma + \xi.$$

.

Therefore $\alpha = \gamma$ implies $\beta = \xi$.

By A.A.S. it follows that $\triangle ABC \cong \triangle BB_1C_1$, and hence $\alpha = \delta = |\angle BC_1 B_1|$. This implies that $\eta + \delta = 90°$. But then it follows that $|\angle DC_1 B_1| = \eta + \delta = 90°$. ■

__Lemma 8B.__ *If there is a rectangle* $\square$**ABCD** *in the neutral piane under consideration, then for every positive integer* n *there is a rectangle* $\square A'B'C'D'$ *with* $|A'B'| = |C'D'| = n|AB| = n|CD|$ *and* $|A'D'| = |B'C'| = |AD| = |BC|$.

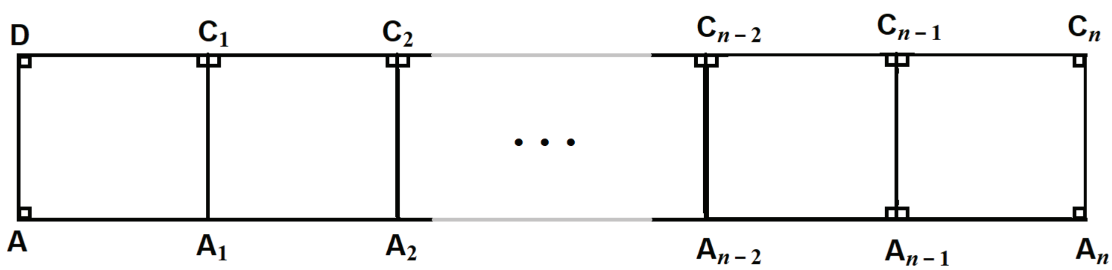
__Proof.__ The case $n = 2$ was done in the preceding lemma. Assume by induction that we have $B = A_1, A_2, \dots, A_{n-1}$ and $C = C_1, C_2, \dots, C_{n-1}$ such that

$$A = A_0 * A_1 * A_2 * \dots * A_{n-1} \quad \text{and} \quad D = C_0 * C_1 * C_2 * \dots * C_{n-1}$$

(the notation means that $X_p * X_q * X_r$ if $p < q < r$) and the following additional conditions:

For $k = 1, \dots, n-1$ we have $|AB| = |CD| = |A_{k-1}A_k| = |C_{k-1}C_k|$.

For $k = 1, \dots, n-1$ the line C_kA_k is perpendicular to both **AB** and **CD**.



Now apply Lemma 8A to the rectangle $A_{n-2}A_{n-1}C_{n-1}C_{n-2}$ to obtain A_n and C_n such that $A_{n-2} * A_{n-1} * A_n$ and $C_{n-2} * C_{n-1} * C_n$, $|AB| = |CD| = |A_{n-1}A_n| = |C_{n-1}C_n|$, and C_nA_n is perpendicular to both **AB** and **CD**. ■

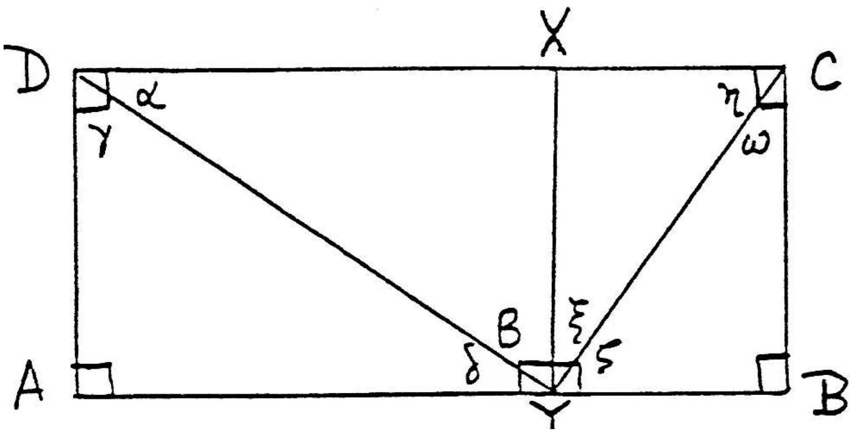
__Corollary 8C.__ *If there is a rectangle* $\square$**ABCD** *in the neutral piane under consideration, then for each pair of integers* $n, m > 0$ *there is a rectangle* $\square A'B'C'D'$ *with* $|A'B'| = |C'D'| = n|AB| = n|CD|$ *and* $|A'D'| = |B'C'| = m|AD| = m|BC|$.

__Proof.__ First apply Lemma B to get a rectangle $\square A^*B^*C^*D^*$ with $|A^*B^*| = n|AB|$ and $|B^*C^*| = |BC|$. Now apply Lemma B again to get a new rectangle $\square A'B'C'D'$ with

$|A'B'| = |A^*B^*|$ and $|B'C'| = m|B^*C^*|$. It follows that $|A'B'| = |C'D'| = n|AB| = n|CD|$ and $|A'D'| = |B'C'| = m|AD| = m|BC|$. ■

The next lemma allows us to take a large rectangle and trim it to another of srnalfer size.

Lemma 8D. *Let* **☐ABCD** *be a rectangle*, *let* **X ∈ (CD),** *and let* **Y** *be the foot of the perpendicular from* **X** *to* **AB.** *Then* **A, Y, X,** *and* **D** *(**in that order**) are the vertices of a rectangle.*



Proof. The lines **AD, XY,** and **BC** are all parallel because they are all perpendicular to **AB.** Hence **AD** is contained in the **D** – side of **XY** and **BC** is contained in the **C** – side of **XY.** But **C∗X∗D** since **X** lies on **(BC)**, and therefore **C** and **D** lie on opposite sides of **XY.** Hence **AD** and **BC** also lie entirely on opposite sides of **XY.** Since **AB** and **XY** meet at **Y,** it follows that **A∗Y∗B** must be true.

Label the angle measures as indicated in the drawing above:

$\alpha = \vert\angle YDX\vert$		$\xi = \vert\angle CYX\vert$
$\beta = \vert\angle DYX\vert$		$\eta = \vert\angle XCY\vert$
$\gamma = \vert\angle ADY\vert$		$\zeta = \vert\angle CYB\vert$
$\delta = \vert\angle AYD\vert$		$\omega = \vert\angle YCB\vert$

Since **AD** is parallel to **XY, CD** is parallel to **XY,** and **AB** is parallel to **CD,** it follows that **A, Y, X,** and **D** and **Y, B, X,** and **X** (in these orders) form tbe vertices of a convex quadrilateral. Therefore **D** lies in the interior of $\angle$**AYX, Y** lies in the interior of $\angle$**ADYX, C** lies in the interior of $\angle$**XYB,** and **Y** lies in the interior of $\angle$**XCB.** These imply the following four equations:

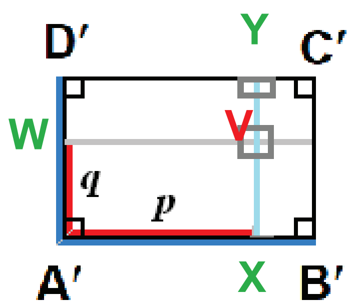
$\alpha + \gamma = 90°$		$\xi + \zeta = 90°$
$\beta + \delta = 90°$		$\eta + \omega = 90°$

The Saccheri – Legendre Theorem implies the following additional inequalities:

$\xi + \gamma \leq 90°$		$\omega + \zeta \leq 90°$

Taken together, these imply $90° \le \alpha + \beta$ and $90° \le \xi + \eta$. Therefore the Saccheri – Legendre Theorem irnplies that both $|\angle \mathbf{DXY}|$ and $|\angle \mathbf{CXY}|$ are less than or equal to $90°$. On the other hand, **C**∗**X**∗**D** implies that $|\angle \mathbf{DXY}| + |\angle \mathbf{CXY}| = 180°$. This can happen only if both $|\angle \mathbf{DXY}|$ and $|\angle \mathbf{CXY}|$ are equal to $90°$. But this now implies **XY** is perpendicular to **CD,** so that **A, Y, X,** and **D** (in that order) *are* the vertices of a rectangle. ■

__**Proof of Theorem 8.**__ Given rectangle ☐**ABCD** and real numbers $p, q > 0$, find positive integers n and m so that $p < n|\mathbf{AB}|$ and $q < m|\mathbf{AD}|$. Form a rectangle ☐**A′B′C′D′** with $|\mathbf{A'B'}| = n|\mathbf{AB}|$ and $|\mathbf{A'D'}| = q|\mathbf{AD}|$.



should be XY

Let $\mathbf{X} \in (\mathbf{A'B'})$ satisfy $|\mathbf{A'X}| = p < |\mathbf{A'B'}|$, and let **Y** be the foot of the perpendicular from **X** to **C′D′.** Then by Lemma 8D one obtains a rectangle ☐**A′~~YX~~D′** with $|\mathbf{A'X}| = p$ and $|\mathbf{A'D'}| = q|\mathbf{AD}|$.

should be X

Now let $\mathbf{W} \in (\mathbf{A'D'})$ satisfy $|\mathbf{A'W}| = q < |\mathbf{A'D'}|$, and let **V** be the foot of the perpendicular from **Z** to **XY.** Then **A′, ~~Y~~, V,** and **W** (in that order) are the vertices of a rectangle with $|\mathbf{A'X}| = p$ and $|\mathbf{A'W}| = q$. ■