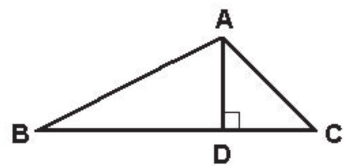
that the angle sum of $\triangle$**WXY** is equal to $180°$, so the same must be true for $\triangle$**ABC.** ■

This result extends directly to arbitrary triangles in the neutral plane $\mathbb{P}$.

Theorem 10. *If a rectangle exists in a neutral plane $\mathbb{P}$, then every triangle in $\mathbb{P}$ has an angle sum equal to $180°$.*

Proof. The idea is simple; we split the given triangle into two right triangles and apply the preceding result. By a corollary to the Exterior Angle Theorem, we know that the perpendicular from one vertex of a triangle meets the opposite side in a point between the other two vertices (in particular, we can take the vertex opposite the longest side). Suppose now that the triangle is labeled $\triangle$**ABC** so that the foot **D** of the perpendicular from **A** to **BC** lies on the open segment **(BC).**



We know that **D** lies in the interior of $\angle$**BAC,** and therefore we have

$$|\angle BAD| + |\angle DAC| = |\angle BAC|.$$

By the previous result on angle sums for right triangles, we also have

$$|\angle BAD| + |\angle ADB| = 90° = |\angle DAC| + |\angle ACD|$$

and if we combine all these equations we find that

$$|\angle ABC| + |\angle BCA| + |\angle CAB| = 180°$$

which is the desired conclusion. ■

There is also a converse to the preceding two results.

Theorem 11. *If a neutral plane $\mathbb{P}$ contains at least one triangle whose angle sum is equal to $180°$, then $\mathbb{P}$ contains a rectangle.*

Proof. The idea is to reverse the preceding discussion; we first show that under the given conditions there must be a right triangle whose angle sum is equal to $180°$, and then we use this to show that there is a rectangle.

FIRST STEP: If there is a triangle whose angle sum is $180°$, then there is also a right triangle with this property.

Given a triangle whose angle sum is $180°$, as in the previous result we label the vertices **A, B, C** so that the foot of the perpendicular from **A** to **BC** lies on the open segment **(BC).** Reasoning once again as in the proof of Theorem 10 we find

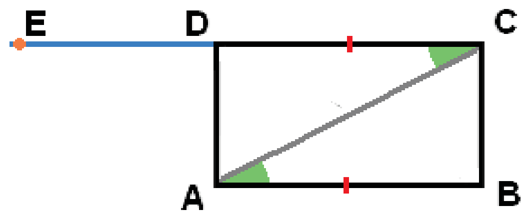
$$\text{Angle sum}(\triangle ABD) + \text{Angle sum}(\triangle ADC) =$$

$$\text{Angle sum}(\triangle ABC) + 180° = 180° + 180° = 360°.$$

Since each of the summands on the left hand side is at most $180°$, it follows that each must be equal to $180°$, (if either were strictly less, then the left side would be less than $360°$). Thus the two right triangles $\triangle$**ABD** and $\triangle$**ADC** have angle sums equal to $180°$.

SECOND STEP: If there is a right triangle whose angle sum is $180°$, then there is also a rectangle.

Once again the idea is simple. We shall construct another right triangle with the same hypotenuse to obtain a rectangle. Suppose that $\triangle$**ABC** is the right triangle whose angle sum is equal to $180°$, and that the right angle of this triangle is at **B.**



See lecture16aa.pdf for a more complete proof

By the Protractor Postulate there is a unique ray **[CE** such that **(CE** is on the side of **AC** opposite **B** and $|\angle$**ECA**$|$ = $|\angle$**BAC**$|$. Take **D** to be the unique point on **(CE** such that $|$**AB**$|$ = $|$**CD**$|$. Then Theorem **7** and S.A.S. imply that $\triangle$**BAC** $\cong$ $\triangle$**DCA.** In particular, we have $|\angle$**DAC**$|$ = $|\angle$**BCA**$|$ and $|\angle$**ADC**$|$ = $|\angle$**ABC**$|$.

It follows that **AD** and **DC** are perpendicular, so we know there are right angles at **B** and **D.** Furthermore, the Alternate Interior Angle Theorem (more correctly, *the half which is valid in neutral geometry*) implies that the lines **AB** and **CD** are parallel, and likewise the same result and the triangle congruence imply that **AD** and **BC** are parallel. As in the discussion of Lambert quadrilaterals, these conditions imply that **A, B, C, D** form the vertices of a convex quadrilateral. We shall use this to prove that there are also right angles at **A** and **C.**

Since we now know we have a convex quadrilateral, it follows that **A** and **C** lie in the interiors of $\angle$**BCD** and $\angle$**DAB** respectively. Therefore we have

$$|\angle BCD| \;=\; |\angle ACD| + |\angle ACB| \;=\; |\angle BAC| + |\angle ACB| \;=\; 90°$$

where the last equation holds because of our assumption about the angle sum of the right triangle $\triangle$**ABC.** Thus we know that there also is a right angle at the vertex **C.** But we also have

$$|\angle BAD| \;=\; |\angle BAC| + |\angle BCA| \;=\; |\angle ACD| + |\angle BCA| \;=\; 90°$$

where the final equation this time follows because we have shown there is a right angle at **C.** Thus we see that there is also a right angle at **A** and therefore we have a rectangle.■

This brings us to the main result of this section.

<u>Theorem 12. (All – or – Nothing Theorem)</u> *In a given neutral plane* $\mathbb{P}$, EITHER ***<u>every triangle</u>*** *has an angle sum is equal to* $180°$ OR ELSE ***<u>no triangle</u>*** *has an angle sum equal to* $180°$. *In the second case the angle sum of every triangle is strictly less than* $180°$.

<u>Proof.</u> This is mainly a matter of sorting through the preceding results. If one triangle has an angle sum equal to $180°$, then by Theorem 11 a rectangle exists, and in that case Theorem 10 implies that every triangle has angle sum equal to $180°$. Therefore it is impossible to have a neutral plane in which some triangles have angle sums equal to $180°$ but others do not. Finally, by the Saccheri – Legendre Theorem we know that if no triangle has angle sum equal to $180°$ then every triangle must have an angle sum that is strictly less than $180°$. ■

The path to hyperbolic geometry

> The sum of the three angles of a plane triangle cannot be greater than $180°$ … But the situation is quite different in the second part — that the sum of the angles cannot be less than $180°$; this is the critical point, the reef on which all the wrecks occur.
>
> C. F. Gauss, ***Letter to F. (W.) Bolyai***
>
> When you have eliminated the impossible, whatever remains, however improbable [it may seem], must be the truth.
>
> A. C. Doyle (1859 – 1930), ***Sherlock Holmes — Sign of the Four***

In some respects, the results of this section provide reasons to be optimistic about finding a proof of Euclid's Fifth Postulate in an arbitrary neutral plane. First of all, the results on rectangles and angle sums show that Playfair's Postulate is equivalent to statements that look much weaker (for example, the existence of ***just one rectangle*** or ***just one triangle whose angle sum is*** $180°$). Furthermore, the results suggest that the negation of Playfair's Postulate leads to consequences which seem extremely strange and perhaps even unimaginable. However, as Gauss indicated in his letter, no one was able to overcome the final hurdle and give a complete proof of Euclid's Fifth Postulate from the other axioms for Euclidean geometry. Although the efforts to prove Euclid's Fifth Postulate did not lead to the proof, the best work on the problem provided very extensive, and in some cases nearly complete, information on strange things that would happen if one assumes that the Fifth Postulate is false. We shall examine some of these phenomena in the remaining sections of this unit.

Ultimately these considerations led to a viewpoint expressed in another quotation from Gauss' correspondence:

> The theorems of this geometry appear to be paradoxical and, to the uninitiated, absurd; but calm, steady reflection reveals that they contain nothing at all impossible. (***Letter to Taurinus***, 1824; one should compare this to the Sherlock Holmes quotation given above.)

Before Gauss, some mathematicians (for example, Klügel) had speculated that a proof of the Fifth Postulate might be out of reach. However, Gauss (and to a lesser extent a contemporaries like Schweikart and Taurinus) took things an important step further, concluding that the negation of the Fifth Postulate yields a geometrical system which is very different from Euclidean geometry in some respects but has ***exactly the same degree of logical validity*** (compare also the passage from the letter to Olbers at the beginning of this unit). Working independently of Gauss, J. Bolyai (1802 – 1860) and N. I. Lobachevsky (1792 – 1856) reached the same conclusions as Gauss (each one independently of the other), which Bolyai summarized in a frequently repeated quotation:

Out of nothing I have created a strange new universe.

Both Bolyai and Lobachevsky took everything one important step further than Gauss by publishing their conclusions, and for this reason they share credit for the first published recognition of hyperbolic geometry as a mathematically legitimate subject.

ADDITIONAL READING FOR THIS SECTION

Sections 10.2 and 10.3 of Moise also contain important material that should be read and well understood. Something from them may appear on an exam.