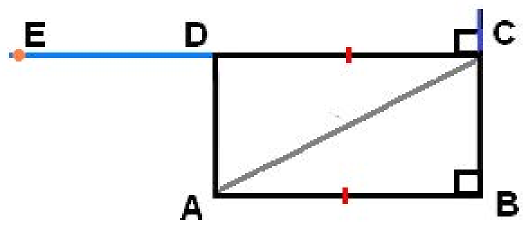
Math 133
Spring 2022
L16AA

ALTERNATE PROOF FOR THEOREM 11

More precisely, this is for the second part of the result: *If there is a right triangle whose angle sum is* $180°$, *then a rectangle exists*.



Let $\triangle ABC$ be the triangle in question, where $|\angle ABC| = 90°$. Next, let E be a point on the same side of BC as A such that $|\angle ECB| = 90°$, and choose $D \in (CE$ so that $|CD| = |AB|$. By Proposition 5 in Lecture 15, the points A, B, C, D (in that order) form the vertices of a convex quadrilateral, and by construction this quadrilateral is a Saccheri quadrilateral with base $[BC]$.

Since the diagonals (AC) and (BD) have a point in common, it follows that A lies in the interior of $\angle DCB$ and consequently

$$90° \quad = \quad |\angle DCA| \ + \ |\angle ACB| \ .$$

On the other hand, since the angle sum of $\triangle ABC$ is $180°$ we also have $|\angle CAB| + |\angle ACB| = 90°$. Combining these equations, we conclude that $|\angle CAB| = |\angle DCA|$. Therefore $\triangle CAB \cong \triangle ACD$ by S.A.S. congruence. The latter shows that $|\angle ADC| = |\angle ABC| = 90°$, and therefore we have $AD \perp CD$.

To complete the proof, we must also show that $|\angle DAB| = 90°$. Since we have a convex quadrilateral, we know that C lies in the interior of $\angle DAB$. Therefore we have

$$|\angle DAB| \quad = \quad |\angle CAB| \ + \ |\angle CAD|$$

and by the triangle congruence and assumptions on $\triangle ABC$ we know that

$$|\angle CAD| \quad = \quad |\angle ACB| \quad = \quad 90° \ - \ |\angle CAB| \ .$$

Therefore $|\angle DAB| = 90°$, so that $AD \perp AB$ and hence the the four points A, B, C, D (in that order) form the vertices of a rectangle.■