4 : Angle defects and related phenomena

In the previous section we showed that the angle sums of triangles in a neutral plane can behave in one of two very distinct ways. In fact, it turns out that there are essentially only two possible neutral planes, one of which is given by Euclidean geometry and the other of which does not satisfy any of the 24 properties listed in Section 2. The purpose of this section is to study some of these properties for a non – Euclidean plane.

Definition. A neutral plane $(\mathbb{P}, \mathcal{L}, d, \alpha)$ is said to be ***hyperbolic*** if Playfair's Parallel Postulate does **not** hold. In other words,

> there is ***some pair*** **(L, X)**, where **L** is a line in $\mathbb{P}$ and **X** is a point not on **L**, for which there are ***at least two lines through*** **X** which are ***parallel to*** **L.**

The study of hyperbolic planes is usually called **HYPERBOLIC GEOMETRY**.

The name "hyperbolic geometry" was given to the subject by F. Klein (1849 – 1925), and it refers to some relationships between the subject and other branches of geometry which cannot be easily summarized here. Detailed descriptions may be found in the references listed below:

C. F. Adler, ***Modern Geometry: An Integrated First Course*** (2^{nd} Ed.). McGraw – Hill, New York, 1967. ISBN: 0–070–00421–8. [see Section **8.5.3**, pp. 219 – 226]

A. F. Horadam, ***Undergraduate Projective Geometry.*** Pergamon Press, New York, 1970. ISBN: 0–080–17479–5. [see pp. 271 – 272]

H. Levy, ***Projective and Related Geometries.*** Macmillan, New York, 1964. ISBN: 0–000–03704–4. [see Chapter **V**, Section **7**]

A complete and rigorous development of hyperbolic geometry is long and ultimately highly nonelementary, and *it requires a significant amount of input from trigonometry, transcendental functions and differential and integral calculus*. We shall discuss one aspect of the subject with close ties to calculus at the end of this section, but we shall only give proofs that involve "elementary" concepts and techniques.

In the previous section we showed that the angle sum of a triangle in a neutral plane is either always equal to $180°$ or always strictly less than $180°$. We shall begin by showing that the second alternative holds in a hyperbolic plane.

Theorem 1. *In a hyperbolic plane* $\mathbb{P}$ *there is a triangle* $\triangle$**ABC** *such that*

$$|\angle CAB| + |\angle ABC| + |\angle ACB| < 180°.$$

By the results of the preceding section, we immediately have several immediate consequences.

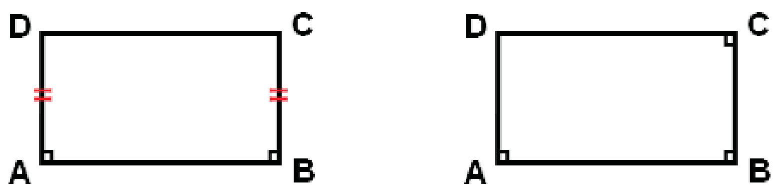
Theorem 2. *In a hyperbolic plane* $\mathbb{P}$*, given an arbitrary triangle* $\triangle$**ABC** *we have*

$$|\angle CAB| + |\angle ABC| + |\angle ACB| < 180°. \blacksquare$$

This follows from the All – or – Nothing Theorem in Section 3, and it has further implications for the near – rectangles we have discussed.

Corollary 3. *In a hyperbolic plane* $\mathbb{P}$*, suppose that we have a convex quadrilateral* $\square$**ABCD** *such that* **AB** *is perpendicular to both* **AD** *and* **BC.**

1. *If* $\square$**ABCD** *is a* ***Saccheri quadrilateral*** *with base* **AB** *such that* $|AD| = |BC|$, *then* $|\angle ADC| = |\angle BCD| < 90°$.
2. *If* $\square$**ABCD** *is a* ***Lambert quadrilateral*** *such that* $|\angle ABC| = |\angle BCD| = |\angle DAB| = 90°$, *then* $|\angle ADC| < 90°$.



In particular, it follows that **there are NO RECTANGLES in a hyperbolic plane** $\mathbb{P}$.

Proof of Corollary 3. If we split each choice of convex quadrilateral into two triangles along the diagonal **[AC]**, then by Theorem 2 we have the following:

$$|\angle CAB| + |\angle ABC| + |\angle ACB| < 180°$$

$$|\angle CAD| + |\angle ADC| + |\angle ACD| < 180°$$

Since is a convex quadrilateral we know that **C** lies in the interior or $\angle$**DAB** and **A** lies in the interior of $\angle$**BCD.** Therefore we have $|\angle DAB| = |\angle DAC| + |\angle CAB|$ and $|\angle BCD| = |\angle ACD| + |\angle ACB|$; if we combine these with the previous inequalities we obtain ***the following basic inequality, which is valid for an arbitrary convex quadrilateral in a hyperbolic plane***:

$$|\angle ABC| + |\angle BCD| + |\angle CDA| + |\angle DAB| =$$

$$|\angle CAB| + |\angle ABC| + |\angle ACB| + |\angle CAD| + |\angle ADC| + |\angle ACD| < 360°$$

To prove the first statement, suppose that $\square$**ABCD** is a ***Saccheri quadrilateral***, so that $|\angle ADC| = |\angle BCD|$ by the results of the previous section. Since $|\angle DAB| = |\angle ABC| = 90°$ by Proposition 3.6, the preceding inequality reduces to

$$180° + |\angle BCD| + |\angle CDA| = 180° + 2|\angle BCD| =$$

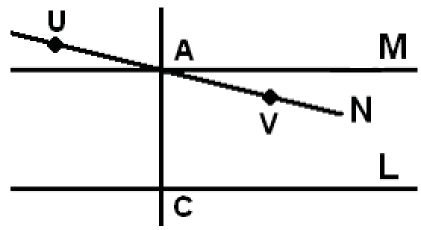
$$180° + 2|\angle CDA| < 360°$$

which implies $|\angle ADC| = |\angle BCD| < 90°$.

To prove the second statement, suppose that $\square$**ABCD** is a ***Lambert quadrilateral***, so that $|\angle BCD| = 90°$. Since $|\angle ABC| = |\angle DAB| = 90°$, the general inequality specializes in this case to $270° + |\angle CDA| < 360°$, which implies the desired inequality $|\angle ADC| < 90°$.■

***Proof of Theorem* 1.** In a hyperbolic plane, we know that there is some line **L** and some point **A** not on **L** such that there are at least two parallel lines to **L** which contain **A.**

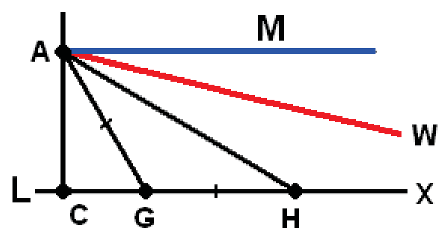
Let **C** be the foot of the unique perpendicular from **A** to **L,** and let **M** be the unique line through **A** which is perpendicular to **AC** in the plane of **L** and **A.** Then we know that **L** and **M** have no points in common (otherwise there would be two perpendiculars to **AC** through some external point). By the choice of **A** and **L** we know that there is a second line **N** through **A** which is disjoint from **L**.



The line **N** contains points **U** and **V** on each side of **AC,** and they must satisfy **U**∗**A**∗**V.** Since **N** is not perpendicular to **AC** and $|\angle \mathbf{CAU}| + |\angle \mathbf{CAV}| = 180°$, it follows that one of $|\angle \mathbf{CAU}|$, $|\angle \mathbf{CAV}|$ must be less than $90°$. Choose **W** to be either **U** or **V** so that we have $\theta = |\angle \mathbf{CAW}| < 90°$ (in the drawing above we have **W** = **V**).

The line **L** also contains points on both sides of **AC,** so let **X** be a point of **L** which is on the same side of **AC** as **W.**

<u>CLAIM:</u> *If* **G** *is a point of* **(CX,** *then there is a point* **H** *on* **(CX** *such that* **C**∗**G**∗**H** *and* $|\angle \mathbf{CHA}| \leq \frac{1}{2}|\angle \mathbf{CGA}|$.



To prove the claim, let **H** be the point on **(CX** such that $|\mathbf{CH}| = |\mathbf{CG}| + |\mathbf{GA}|$; it follows that **C**∗**G**∗**H** holds and also that $|\mathbf{GH}| = |\mathbf{AG}|$. The Isosceles Triangle Theorem then implies that $|\angle \mathbf{GHA}| = |\angle \mathbf{GAH}|$, and by a corollary to the Saccheri – Legendre Theorem we also have $|\angle \mathbf{CGA}| \geq |\angle \mathbf{GHA}| + |\angle \mathbf{GAH}| = 2|\angle \mathbf{GHA}| = 2|\angle \mathbf{CHA}|$, where the final equation holds because $\angle \mathbf{GHA} = \angle \mathbf{CHA}$. ***<u>This proves the claim.</u>***

Proceeding inductively, we obtain a sequence of points $\mathbf{B}_0, \mathbf{B}_1, \mathbf{B}_2, \ldots$ of points on **(CH** such that $|\angle \mathbf{CB}_{k+1}\mathbf{A}| \leq \frac{1}{2}|\angle \mathbf{CB}_k\mathbf{A}|$, and it follows that for each n we have

$$|\angle \mathbf{CB}_n\mathbf{A}| \leq 2^{-n}|\angle \mathbf{CB}_0\mathbf{A}|.$$

If we choose n large enough, we can make the right hand side (hence the left hand side) of this inequality less than $\frac{1}{2}(90° - \theta)$. Furthermore, we can also choose n so that

$$|\angle CB_nA| \;<\; \theta \;=\; |\angle CAW|$$

and it follows that the angle sum for $\triangle AB_nC$ will be

$$|\angle CAB_n| + |\angle AB_nC| + |\angle ACB_n| \;<$$
$$\tfrac{1}{2}(180° - \theta) + \theta + 180° \;<\; (90° - \theta) + \theta + 90° \;=\; 180°.$$

Therefore we have constructed a triangle whose angle sum is less than $180°$, as required.■

Definition. Given $\triangle ABC$ in a hyperbolic plane, its ***angle defect*** is given by

$$\delta(\triangle ABC) \;=\; 180° - |\angle CAB| - |\angle ABC| - |\angle ACB|.$$

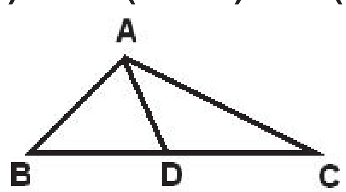
By Theorem 2, ***in a hyperbolic plane the angle defect of $\triangle ABC$ is a positive real number which is always strictly between $0°$ and $180°$.***■

The Hyperbolic Angle – Angle – Angle Congruence Theorem

We have already seen that in spherical geometry there is a complementary notion of ***angle excess***, and the area of a spherical triangle is proportional to its angle excess. There is a similar phenomenon in hyperbolic geometry: ***For any geometrically reasonable theory of area in hyperbolic geometry, the angle of a triangle is proportional to its angular defect.*** This is worked out completely in the book by Moïse. However, for our purposes we only need the following property which suggests that the angle defect behaves like an area function.

Proposition 4. (Additivity property of angle defects) *Suppose that we are given $\triangle ABC$ and that D is a point on (BC). Then we have*

$$\delta(\triangle ABC) \;=\; \delta(\triangle ABD) + \delta(\triangle ADC).$$



Proof. If we add the defects of the triangles we obtain the following equation:

$$\delta(\triangle ABD) + \delta(\triangle ADC) \;=\; 180° - |\angle DAB| - |\angle ABD| - |\angle ADB| +$$
$$180° - |\angle CAD| - |\angle ADC| - |\angle ACD|$$

By the Supplement Postulate for angle measure we know that

$$|\angle ADB| + |\angle ADC| \;=\; 180°$$

by the Additivity Postulate we know that

$$|\angle BAC| = |\angle BAD| + |\angle DAC|$$

and by the hypotheses we also know that $\angle ABD = \angle ABC$ and $\angle ACD = \angle ACB$. If we substitute all these into the right hand side of the equation for the defect sum $\delta(\triangle ABD) + \delta(\triangle ADC)$, we see that this right hand side reduces to

$$180^\circ - |\angle CAB| - |\angle ABC| - |\angle ACB|$$

which is the angle defect for $\triangle ABC$.■

The next result yields a striking conclusion in hyperbolic geometry, which shows that ***the latter does not have a similarity theory comparable to that of Euclidean geometry.***

Theorem 5. (Hyperbolic A.A.A. or *Angle – Angle – Angle Congruence Theorem*) *Suppose we have ordered triples* (A, B, C) *and* (D, E, F) *of noncollinear points such that the triangles* $\triangle ABC$ *and* $\triangle DEF$ *satisfy* $|\angle CAB| = |\angle FDE|$, $|\angle ABC| = |\angle DEF|$, *and* $|\angle ACB| = |\angle DFE|$. *Then we have* $\triangle ABC \cong \triangle DEF$.

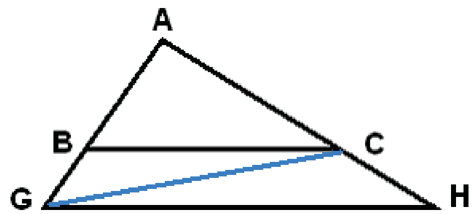
Lecture 16 ends here.

Proof. If at least one of the statements $|BC| = |EF|$, $|AB| = |DE|$, or $|AC| = |DF|$ is true, then by **A.S.A.** we have $\triangle ABC \cong \triangle DEF$. Therefore it is only necessary to consider possible situations in which all three of these statements are false. This means that in each expression, one term is less than the other. There are eight possibilities for the directions of the inequalities, and these are summarized in the table below.

This should be $|DF|$.

CASE	$\lvert AB\rvert$?? $\lvert DE\rvert$	$\lvert AC\rvert$?? /DF/	$\lvert BC\rvert$?? $\lvert EF\rvert$
000	$\lvert AB\rvert < \lvert DE\rvert$	$\lvert AC\rvert < \lvert DF\rvert$	$\lvert BC\rvert < \lvert EF\rvert$
001	$\lvert AB\rvert < \lvert DE\rvert$	$\lvert AC\rvert < \lvert DF\rvert$	$\lvert BC\rvert > \lvert EF\rvert$
010	$\lvert AB\rvert < \lvert DE\rvert$	$\lvert AC\rvert > \lvert DF\rvert$	$\lvert BC\rvert < \lvert EF\rvert$
011	$\lvert AB\rvert < \lvert DE\rvert$	$\lvert AC\rvert > \lvert DF\rvert$	$\lvert BC\rvert > \lvert EF\rvert$
100	$\lvert AB\rvert > \lvert DE\rvert$	$\lvert AC\rvert < \lvert DF\rvert$	$\lvert BC\rvert < \lvert EF\rvert$
101	$\lvert AB\rvert > \lvert DE\rvert$	$\lvert AC\rvert < \lvert DF\rvert$	$\lvert BC\rvert > \lvert EF\rvert$
110	$\lvert AB\rvert > \lvert DE\rvert$	$\lvert AC\rvert > \lvert DF\rvert$	$\lvert BC\rvert < \lvert EF\rvert$
111	$\lvert AB\rvert > \lvert DE\rvert$	$\lvert AC\rvert > \lvert DF\rvert$	$\lvert BC\rvert > \lvert EF\rvert$

Reversing the roles of the two triangles if necessary, we may assume that at least two of the sides of $\triangle ABC$ are shorter than the corresponding sides of $\triangle DEF$. Also, if we consistently reorder $\{A, B, C\}$ and $\{D, E, F\}$ in a suitable manner, then we may also arrange things so that $|AB| < |DE|$ and $|AC| < |DF|$. Therefore, if we take points G and H on the respective open rays $(BA$ and $(BC$ such that $|AG| = |DE|$ and $|AH| = |DF|$, then by **S.A.S.** we have $\triangle AGH \cong \triangle DEF$.



By hypothesis and construction we know that the angular defects of these triangles satisfy $\delta(\triangle\mathbf{AGH}) = \delta(\triangle\mathbf{DEF}) = \delta(\triangle\mathbf{ABC})$. We shall now derive a contradiction using the additivity property of angle defects obtained previously. The distance inequalities in the preceding paragraph imply the betweenness statements $\mathbf{A*B*G}$ and $\mathbf{A*C*H}$, which in turn yield the following defect equations:

$$\delta(\triangle\mathbf{AGH}) = \delta(\triangle\mathbf{AGC}) + \delta(\triangle\mathbf{GCH})$$

$$\delta(\triangle\mathbf{AGC}) = \delta(\triangle\mathbf{ABC}) + \delta(\triangle\mathbf{BGC})$$

If we combine these with previous observations and the positivity of the angle defect we obtain

$$\delta(\triangle\mathbf{ABC}) < \delta(\triangle\mathbf{ABC}) + \delta(\triangle\mathbf{BGC}) + \delta(\triangle\mathbf{GCH}) = \delta(\triangle\mathbf{AGH}) = \delta(\triangle\mathbf{DEF})$$

which contradicts the previously established equation $\delta(\triangle\mathbf{DEF}) = \delta(\triangle\mathbf{ABC})$. The source of this contradiction is our assumption that the corresponding sides of the two triangles do not have equal lengths, and therefore this assumption must be false. As noted at the start of the proof, this implies $\triangle\mathbf{ABC} \cong \triangle\mathbf{DEF}$.■

One immediate consequence of Theorem 6 is that ***in hyperbolic geometry, two triangles cannot be similar in the usual sense unless they are congruent.*** In particular, this means that we cannot magnify or shrink a figure in hyperbolic geometry without distortions. This is disappointing in many respects, but if we remember that angle defects are supposed to behave like area functions then this is not surprising; we expect that two similar but noncongruent figures will have different areas, and in hyperbolic (just as in spherical!) geometry this simply cannot happen.

The Strong Hyperbolic Parallelism Property

The negation of Playfair's Postulate is that there is ***some line*** and ***some external point*** for which ***parallels are not unique.*** It is natural to ask if there are neutral geometries in which unique parallels exist for ***some but not all*** pairs **(L, A)** where **L** is a line and **A** is an external point. The next result implies that no such neutral geometries exist.

Theorem 7. *Suppose we have a neutral plane* $\mathbb{P}$ *such that for* ***some*** *line* **L** *and* ***some external point*** **A** *there is a unique parallel to* **L** *through* **A***. Then there is a rectangle in* $\mathbb{P}$*.*