Math 133
Fall 2021

SOLUTIONS FOR "MORE WEEK 08 EXERCISES"

11. (a) The midpoint conditions imply the following equations:

$$|AE| \ = \ |EC| \ = \ |AC|/2 \ = \ |A'C'|/2 \ = \ = \ |E'C'| \ = \ |A'E'|$$

$$|AF| \ = \ |FC| \ = \ |AB|/2 \ = \ |A'B'|/2 \ = \ = \ |F'B'| \ = \ |A'F'|$$

$$|BD| \ = \ |DC| \ = \ |BC|/2 \ = \ |B'C'|/2 \ = \ = \ |D'C'| \ = \ |B'D'|$$

Furthermore, we are given that $|\angle CAB = \angle EAF| = |\angle C'A'B' = \angle E'A'F'|$, $|\angle ABC = \angle FBD| = |\angle A'B'C' = |\angle F'B'D'|$, and $|\angle ACB = \angle ECD| = |\angle A'C'B' = \angle E'C'D'|$. so by SAS we have the congruences $\triangle EAF \cong \triangle E'A'F'$, $\triangle FBD \cong \triangle F'B'D'$, and $\triangle ECD \cong \triangle E'C'D'$.■

(a) The triangle congruences in (a) imply that $|EF| = |E'F'|$, $|DF| = |D'F'|$ and $|DE| = |D'E'|$. Therefore we also have $\triangle DEF \cong \triangle D'E'F'$ by SSS.■

(c) Consider $\triangle ABC$ and $\triangle AFE$ first. By SAS similarity we have $\triangle AFE \sim \triangle ABC$ with ratio of similitude equal to $\frac{1}{2}$. Therefore $|EF| = |BC|/2$. Similarly $|E'F'| = |B'C'|/2$. Interchanging the roles of A, B, C and D, E, F (and the corresponding primed vertices) in a compatible manner consistent with the midpoint notation, we likewise concludde that $|DF| = |AC|/2$, $|D'F'| = |A'C'|/2$, $|DE| = |AB|/2$ and $|D'E'| = |A'B'|/2$. Combining this with $\triangle ABC \cong \triangle A'B'C'$, by SSS congruence we obtain

$$\triangle AEF \ \cong \ \triangle FDB \ \cong \ \triangle CED \ \cong \ \triangle DFE \ \cong$$

$$\triangle A'E'F' \ \cong \ \triangle F'D'B' \ \cong \ \triangle C'E'D' \ \cong \ \triangle D'F'E'$$

which is what we wanted to prove; in subsequent exercises we shall see that the analogous result in hyperbolic geometry is false.■

12. (a) Suppose first that we have a Saccheri quadrilateral $\diamondsuit ABCD$ in a hyperbolic plane with base $[AB]$. By a theorem in Section 16.3 of Moise, we know that $|AB| \leq |CD|$, and furthermore by a previous exercise we know that if the Saccheri quadrilateral is a rectangle if equality holds. Since rectangles do not exist in a hyperbolic plane, we must have the strict inequality $|AB| < |CD|$.

Now suppose that that we have a Lambert quadrilateral $\diamondsuit ABCD$ in a hyperbolic plane with right angles at A, B, C. By Exercise V.3.9 and V.3.10 we know that $d(A, B) \leq d(C, D)$ and $d(A, D) \leq d(B, C)$, and if either $d(A, B) = d(C, D)$ or $d(A, D) = d(B, C)$ then the Lambert quadrilateral is a rectangle. As above, since rectangles do not exist in a hyperbolic plane, we must have the strict inequalities $d(A, B) < d(C, D)$ and $d(A, D) < d(B, C)$.■

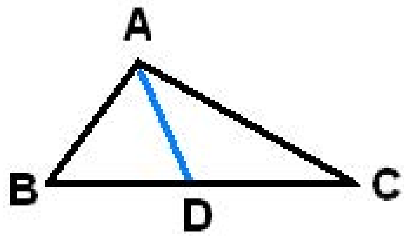
(b) This follows fairly directly from results in Section 4 of the notes. By an exercise from the preceding section, we know that the lines containing the summit and base of the Saccheri quadrilateral have a common perpendicular, and the theorem from the notes says that the shortest distance from a point on one line to the other is realized at the points where the two parallel lines meet this common perpendicular. Since the lines containing the lateral sides of a Saccheri quadrilateral are perpendicular to the line containing the base, it follows that the length of a lateral side must be greater than the length of the segment joining the midpoints of the summit and base, for the line joining these two points is the common perpendicular.■

(c) In a Saccheri quadrilateral both summit angles are acute and have the same angular measure. The first assertion follows because the angle sum of a convex quadrilateral in hyperbolic geometry is always less than 360°. In contrast, a Lambert quadrilateral has three right angles at the vertices, and only the remaining vertex angle can be acute.■

13. If we split a triangle ΔABC into two triangles by a segment $[BD]$ where $D \in (AC)$, then we have

$$\delta(\triangle ABC) \;=\; \delta(\triangle ABD) \;+\; \delta(\triangle ADC)$$

and since all numbers in sight are positive it follows that at least one of the numbers on the right hand side is less than or equal to $\frac{1}{2}\delta(\triangle ABC)$.

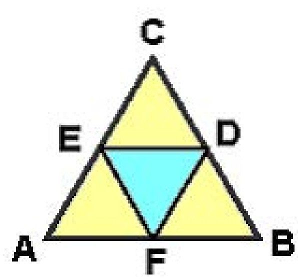


The preceding argument shows that if we are given $\triangle ABC$ then there is some triangle $\triangle X_1Y_1Z_1$ such that $\delta(\Delta X_1Y_1Z_1) \leq \frac{1}{2}\delta(\Delta ABC)$. Repeating this process, for each n we can construct a triangle $\triangle X_nY_nZ_n$ such that $\delta(\Delta X_nY_nZ_n) \leq \delta(\Delta ABC)/2^n$. One can now use the Archimedian Property to show there is some n for which the right hand side is less than h.■

14. (a) As in the proof of the Hyperbolic AAA Congruence Theorem we know that the defects satisfy $\delta(\Delta ADE) < \delta(\Delta ABC)$. If we apply the Isosceles Triangle Theorem and the definition of defect to both triangles we find that $180 - |\angle BAC| - 2|\angle ADE| \;=\; \delta(\Delta ADE) \;<\; \delta(\Delta ABC) \;=\; 180 - |\angle BAC| - 2|\angle ABC|$ and from this point one can use standard manipulations with inequalities to prove that $|\angle ADE| > |\angle ABC|$.■

The file solutions16a.pdf has a drawing for 14(*a*)

(b) Since equilateral triangles are equiangular, we know that $|\angle BAC| = |\angle ABC| = |\angle BCA|$; let us denote this common value by ξ. Since D, E and F are midpoints of the sides of an equilateral triangle, we know that $|AF| \;=\; |FB| \;=\; |BD| \;=\; |DC| \;=\; |CE| \;=\; |EA|$ and therefore we have $\Delta AEF \cong \Delta BFD \cong \Delta CDE$ by **SAS**.



.All three of these smaller triangles are isosceles, so that we also have

$$|\angle AEF| \;=\; |\angle AFE| \;=\; |\angle BFD| \;=\; |\angle BDF| \;=\; |\angle CDE| \;=\; |\angle CED|$$

and we shall denote the common value by η.

The triangle congruences also imply

$$|EF| \quad = \quad |FD| \quad = \quad |DE|$$

and hence ΔDEF is also an equilateral triangle. Thus it is also equiangular, so let φ be the measure of the three vertex angles. The second relationship to proved in the exercise then translates to showing that $\varphi > \xi$.

Since we are working in hyperbolic geometry we know that the angle sum of, say, ΔAEF is less than 180 degrees, and if we substitute the values ξ and η into this inequality we find that $\xi + 2\eta < 180$.

A picture suggests that we should also have $\varphi + 2\eta = 180$, but we need to prove this. A key step in doing this is to show that E lies in the interior of $\angle DFA$. To prove this, first observe that the betweenness relations $C * E * A$ and $C * D * B$ imply that C, D and E all lie on the same side of AB. Next, the betweenness relations $A * F * B$ and $C * D * B$ imply that B lies on the side of FD opposite both C and A, so that A and C lie on the same side of DF. Finally, $E \in (AC)$ now implies that A and E must lie on the same side of DF, completing the requirements for E to lie in the interior of $\angle DFA$.

The preceding paragraph implies that $|\angle DFA| = |\angle DFE| + |\angle EFA| = \varphi + \eta$. Since $A * F * B$ holds, we also have

$$180 \quad = \quad |\angle DFA| \; + \; |\angle DFB| \quad = \quad \varphi \; + \; \eta \; + \; \eta \quad = \quad \varphi \; + \; 2 \cdot \eta$$

which was the claim at the beginning of the preceding paragraph. It now follows that

$$\xi \; + \; 2 \cdot \eta \quad < \quad 180 \quad = \quad \varphi \; + \; 2 \cdot \eta$$

which implies $\xi < \eta$, proving the inequality stated in the second assertion of the exercise.

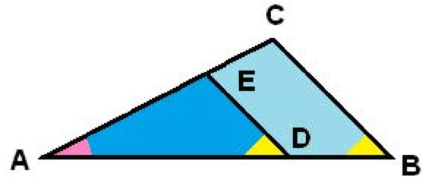
Finally, we need to show that the isosceles triangle ΔAEF is not an equilateral triangle. However, the preceding exercise implies that

$$|\angle EFA| \quad > \quad |\angle ABC|$$

and since the right hand side is equal to $|\angle CAB = \angle EAF$, we can use "the larger angle is opposite the longer side" to conclude that $|AE| < |FA|$.■

15. We know that there is a ray $[DX$ such that $(DX$ lies on the same side of AB as C and $|\angle EDA| = |\angle CBA|$. The rays $[DX$ and $[BC$ cannot have a point in common, for if they met at some point Y then the Exterior Angle Theorem would imply $|\angle EDA| > |\angle CBA|$ and by construction these two numbers are equal.

By Pasch's Theorem the line DX must have a point in common with either $[BC]$ or (AC). Since $[DX$ and $[BC$ have no points in common by the preceding paragraph, it follows that there must be a point $E \in (AC) \cap DX$. Since $A * E * C$ is true, it follows that E and C lie on the same side of AB, so that $[DE = [DX$.



Since $A * E * C$ is true, it follows that E and C lie on the same side of AB, so that $[DE = [DX$. Furthermore, since $E \in (AC)$ and $D \in (AB)$, the angle defects of ΔABC and ΔADE satisfy

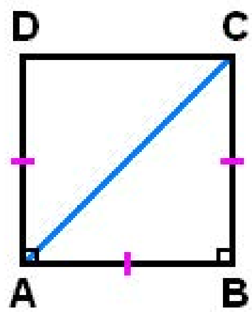
$$\delta(\Delta ABC) \ = \ \delta(\Delta ADE) \ + \ \delta(\Delta EDC) \ + \ \delta(\Delta DBC)$$

so that $\delta(\Delta ADE) < \delta(\Delta ABC)$. On the other hand, by construction we have

$$\delta(\Delta ABC) \ - \ \delta(\Delta ADE) \ = \ |\angle AED| \ - \ |\angle ACB|$$

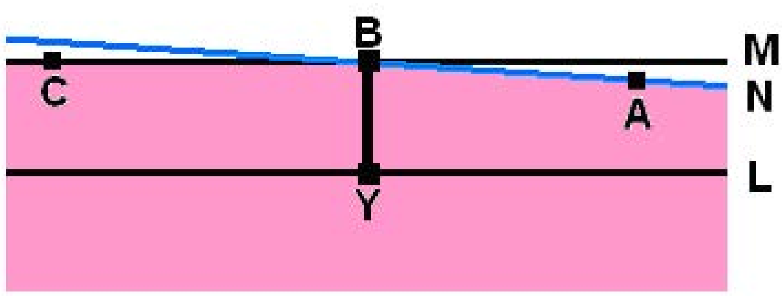
and since the left hand side is positive it follows that $|\angle AED| > |\angle ACB|$, which is what we wanted to prove.■

16. Suppose that the ray $[AC$ bisects $\angle DAB$. Then we have $|\angle CAD| = |\angle DAB| = 45°$.



On the other hand, since ΔABC is an isosceles triangle with a right angle at B, it will follow that $|\angle ACB| = 45°$. In particular, this means that the angle defect of ΔABC is zero. This cannot happen in a hyperbolic plane, and therefore the ray $[AC$ cannot bisect $\angle DAB$.■

17. Follow the hint, so that B is a point not on a line L such that there are at least two parallel lines to L through B. One of the lines can be constructed by dropping a perpendicular from B to L whose foot we shall call Y, and then taking a line M which is perpendicular to BY and passes through B. Let N be a second line through B which is parallel to L.



Since L and M are parallel, all points of L lie on the same side of M. Since N contains points on both sides of M, it follows that there is some point A which lie on N and also on the same side of M as L. Note that $A \notin BY$, because $N \cap BY = \{B\}$ and $B \in M$. Since M contains points on both sides of BY, there is also a point $C \in M$ which lies on the side of BY which does not contain A (hence A and C lie on opposite sides of BY).

We claim that L is contained in the interior of $\angle ABC$. The first step is to show that Y lies in the interior of this angle. By construction we know that $Y \in L$ and since L and A lie on the same side of M, it follows that Y and A lie on the same side of $M = BC$. On the other hand, since A and C lie on opposite sides of BY we know there is a point $Z \in (AC) \cap BY$. It follows that A and Z lie on the same side of $BC = M$, and since A and Y also lie on the same side of M it follows that $(BY = (BZ$. But this means that C, Z and Y must all lie on the same side of $N = AB$. Thus we have shown that Y lies in the interior of $\angle ABC$.

Since L does not have any points in common with either M or N, it follows that all points of L lie on the same side of each line. We have seen that $Y \in L$ lies on the same side of $M = BC$ as A and on the same side of $N = AB$ as C, and therefore the same must be true for every point of L. But this means that L is contained in the interior of $\angle ABC$.■

(b) The location of the line L is arbitrary, so it is useful to begin by disposing of a special case first. If L contains the vertex B, then $B \notin \text{Int}\, ABC$ and we are done. Assume henceforth that $B \notin L$.

We know that the lines AB and BC are distinct, so at most one of them is parallel to L; let M be a line in $\{AB, AC\}$ which is not parallel to L. Then L must contain a point of AB or AC. Since both of these lines are disjoint from $\text{Int}\, ABC$ it follows that L must contain a point which is not in the interior of the angle.■