$$|\angle BAC| = |\angle BAD| + |\angle DAC|$$

and by the hypotheses we also know that $\angle ABD = \angle ABC$ and $\angle ACD = \angle ACB$. If we substitute all these into the right hand side of the equation for the defect sum $\delta(\triangle ABD) + \delta(\triangle ADC)$, we see that this right hand side reduces to

$$180^\circ - |\angle CAB| - |\angle ABC| - |\angle ACB|$$

which is the angle defect for $\triangle ABC$.■

The next result yields a striking conclusion in hyperbolic geometry, which shows that ***the latter does not have a similarity theory comparable to that of Euclidean geometry.***

==

Theorem 5. (Hyperbolic A.A.A. or *Angle – Angle – Angle Congruence Theorem*) *Suppose we have ordered triples* **(A, B, C)** *and* **(D, E, F)** *of noncollinear points such that the triangles* $\triangle ABC$ *and* $\triangle DEF$ *satisfy* $|\angle CAB| = |\angle FDE|$, $|\angle ABC| = |\angle DEF|$, *and* $|\angle ACB| = |\angle DFE|$. *Then we have* $\triangle ABC \cong \triangle DEF$.

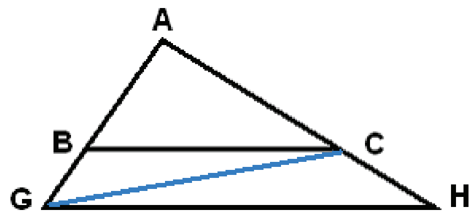
Lecture 17 starts here.

Proof. If at least one of the statements $|BC| = |EF|$, $|AB| = |DE|$, or $|AC| = |DF|$ is true, then by **A.S.A.** we have $\triangle ABC \cong \triangle DEF$. Therefore it is only necessary to consider possible situations in which all three of these statements are false. This means that in each expression, one term is less than the other. There are eight possibilities for the directions of the inequalities, and these are summarized in the table below.

Should be |DF|

CASE	|AB| ?? |DE|	|AC| ?? /DF/	|BC| ?? |EF|
000	$\|AB\| < \|DE\|$	$\|AC\| < \|DF\|$	$\|BC\| < \|EF\|$
001	$\|AB\| < \|DE\|$	$\|AC\| < \|DF\|$	$\|BC\| > \|EF\|$
010	$\|AB\| < \|DE\|$	$\|AC\| > \|DF\|$	$\|BC\| < \|EF\|$
011	$\|AB\| < \|DE\|$	$\|AC\| > \|DF\|$	$\|BC\| > \|EF\|$
100	$\|AB\| > \|DE\|$	$\|AC\| < \|DF\|$	$\|BC\| < \|EF\|$
101	$\|AB\| > \|DE\|$	$\|AC\| < \|DF\|$	$\|BC\| > \|EF\|$
110	$\|AB\| > \|DE\|$	$\|AC\| > \|DF\|$	$\|BC\| < \|EF\|$
111	$\|AB\| > \|DE\|$	$\|AC\| > \|DF\|$	$\|BC\| > \|EF\|$

Reversing the roles of the two triangles if necessary, we may assume that at least two of the sides of $\triangle ABC$ are shorter than the corresponding sides of $\triangle DEF$. Also, if we consistently reorder $\{A, B, C\}$ and $\{D, E, F\}$ in a suitable manner, then we may also arrange things so that $|AB| < |DE|$ and $|AC| < |DF|$. Therefore, if we take points **G** and **H** on the respective open rays **(BA** and **(BC** such that $|AG| = |DE|$ and $|AH| = |DF|$, then by **S.A.S.** we have $\triangle AGH \cong \triangle DEF$.



By hypothesis and construction we know that the angular defects of these triangles satisfy $\delta(\triangle AGH) = \delta(\triangle DEF) = \delta(\triangle ABC)$. We shall now derive a contradiction using the additivity property of angle defects obtained previously. The distance inequalities in the preceding paragraph imply the betweenness statements $A*B*G$ and $A*C*H$, which in turn yield the following defect equations:

$$\delta(\triangle AGH) = \delta(\triangle AGC) + \delta(\triangle GCH)$$

$$\delta(\triangle AGC) = \delta(\triangle ABC) + \delta(\triangle BGC)$$

If we combine these with previous observations and the positivity of the angle defect we obtain

$$\delta(\triangle ABC) < \delta(\triangle ABC) + \delta(\triangle BGC) + \delta(\triangle GCH) =$$
$$\delta(\triangle AGH) = \delta(\triangle DEF)$$

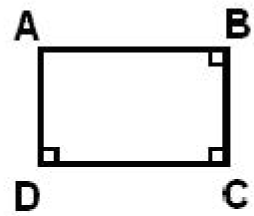
which contradicts the previously established equation $\delta(\triangle DEF) = \delta(\triangle ABC)$. The source of this contradiction is our assumption that the corresponding sides of the two triangles do not have equal lengths, and therefore this assumption must be false. As noted at the start of the proof, this implies $\triangle ABC \cong \triangle DEF$.■

One immediate consequence of Theorem 6 is that ***in hyperbolic geometry, two triangles cannot be similar in the usual sense unless they are congruent.*** In particular, this means that we cannot magnify or shrink a figure in hyperbolic geometry without distortions. This is disappointing in many respects, but if we remember that angle defects are supposed to behave like area functions then this is not surprising; we expect that two similar but noncongruent figures will have different areas, and in hyperbolic (just as in spherical!) geometry this simply cannot happen.

The Strong Hyperbolic Parallelism Property

The negation of Playfair's Postulate is that there is ***some line*** and ***some external point*** for which ***parallels are not unique.*** It is natural to ask if there are neutral geometries in which unique parallels exist for ***some but not all*** pairs **(L, A)** where **L** is a line and **A** is an external point. The next result implies that no such neutral geometries exist.

Theorem 7. *Suppose we have a neutral plane* $\mathbb{P}$ *such that for* ***some*** *line* **L** *and* ***some external point*** **A** *there is a unique parallel to* **L** *through* **A***. Then there is a rectangle in* $\mathbb{P}$*.*



Proof. Let **D** be the foot of the perpendicular from **A** to **L,** and let **C** be a second point on **L.** Let **M** be the line in the plane of **L** and **A** such that **M** is perpendicular to **L** at **C.** Then **AD** and **M** are lines perpendicular to **L** and meet the latter at different points, so that **AD** and **M** are parallel. Next let **B** be the foot of the perpendicular to **M** from the external point **A.** The lines **AB** and **L** are distinct since **A** does not lie on **L,** and since they are both perpendicular to **M** it follows that **AB** and **L** are also parallel. Since we have **AB || CD** and **AD || BC,** it follows that **A, B, C, D** are the vertices of a convex quadrilateral.

If **N** is the perpendicular to **AD** through the point **A** in the plane of **L** and **A,** then we know that **N** is also parallel to **L.** Therefore the uniqueness of parallels to **L** through **A** implies that **N** must be equal to the line **AB** constructed in the previous paragraph; thus we know that **AB** is perpendicular to **AD,** and therefore it follows that the convex quadrilateral ☐**ABCD** is a rectangle.■

Corollary 8. *If* $\mathbb{P}$ *is a neutral plane such that for* ***some line*** **L** *and* ***some external point*** **A** *there is a unique parallel to* **L** *through* **A***, then Playfair's Postulate is true in* $\mathbb{P}$*.*

Proof. By the theorem and the results of the previous section, we know that the angle sum for every triangle in $\mathbb{P}$ is equal to $180°$. On the other hand, if Playfair's Postulate does not hold in $\mathbb{P}$, then by Theorem 2 we know that the angle sum for every triangle is less than $180°$. Therefore Playfair's Postulate must hold in $\mathbb{P}$; in other words, for every line **M** and external point **B** there is a unique parallel to **M** through **B.**■

Asymptotic parallels

We have already noted that Playfair's Postulate is equivalent to the following statement:

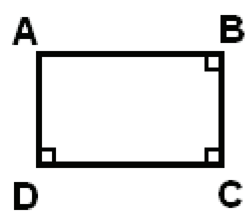
EQUIDISTANCE OF PARALLELS. *Let* **L** *and* **M** *be parallel lines in a neutral plane* $\mathbb{P}$, *let* **X** *be a point on one of the lines, and let* **Y(X)** *be the foot of the perpendicular from* **X** *to the other line. Then the distance* η**(X)** *from* **X** *to* **Y(X)** *is the same for all choices of* **X.**

It is natural to ask what can be said about the distance function η**(X)** if **L** and **M** are parallel lines in a hyperbolic plane $\mathbb{P}$*.* Thus far all of our ***explicit*** examples of parallel lines in hyperbolic planes have been pairs for which there is a common perpendicular (although we have not necessarily proven this in all cases). Our next result describes the behavior of η**(X)** for such pairs of parallel lines.

Theorem 9. *Let* **L** *and* **M** *be parallel lines in a hyperbolic plane* $\mathbb{P}$, *and suppose that* **L** *and* **M** *have a common perpendicular. Then* **L** *and* **M** *have a unique perpendicular, and if* **C** *and* **B** *are points of* **L** *and* **M** *such that* **BC** *is perpendicular to both lines, then the minimum value of* η *is realized at* **C** *and* **B.**

In other words, the distance between two such lines behaves somewhat like the distance between two skew lines in Euclidean $3-$space.

Proof. Let **A** be a second point of **M,** and let **D** be the foot of the perpendicular from **A** to **L;** then **D** and **C** are distinct (otherwise **A, B, C** are collinear, so that **M** = **BC,** which is impossible since **M** is parallel to **L** and **C** lies on **L**), and the four points **A, B, C, D** form the vertices of a Lambert quadrilateral with perpendicular sides at the vertices **B, C** and **D.** By the results and exercises on neutral geometry from the previous section, we have $|BC| \leq |AC|$.



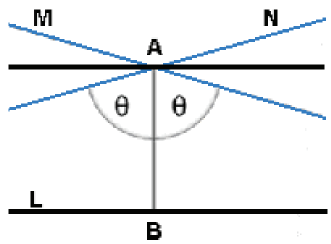
We claim that in fact $|BC| < |AD|$. If equality held, then by **S.A.S.** we would have $\triangle ADC \cong \triangle BCD$. This in turn would imply $|AC| = |BD|$, which would further imply $\triangle DAB \cong \triangle CBA$ by **S.S.S.,** so that $|\angle DAB| = |\angle CBA| = 90°$. Thus the Lambert quadrilateral is a rectangle, and since rectangles do not exist in a hyperbolic plane we have a contradiction. Therefore we must have strict inequality as claimed, and accordingly the shortest distance between the two lines is the distance between **B** and **C** on the common perpendicular.■

For the remainder of this section we shall merely describe the behavior of the function $\eta(X)$ which gives the distance between a point **X** of a line **L** and a line **M** which is parallel to **L.** Further details are in Sections 24.1 – 24.4 of Moise. A major reason for omitting the proofs is that they depend upon the concept of a ***least upper bound*** for a nonempty set of real numbers which is bounded from above; this concept is defined and studied in Chapter 20 of Moise.

Not every pair of parallel lines in a hyperbolic plane has a common perpendicular. Those pairs which have no common perpendicular are an important class sometimes called ***asymptotic parallels.*** For such pairs the function η does not reach a minimum value but can be made less than an arbitrarily small positive real number (hence the lines approach each other asymptotically much as the hyperbola $y = 1/x$ asymptotically approaches the $x-$axis defined by $y = 0$). To describe such lines, suppose that **(L, A)** is a pair consisting of a line **L** in a hyperbolic plane $\mathbb{P}$ and a point **A** which is in $\mathbb{P}$ but not on **L,** and let **B** be the foot of the perpendicular to from

A to **L.** We then have the following result, which is obtainable by combining several separate theorems in Sections 24.1 – 24.4 of Moïse:

<u>**Theorem 10.**</u> *In a hyperbolic plane* $\mathbb{P}$*, let* **L** *be a line, let* **A** *be a point not on* **L**, *and let* **B** *be the foot of the perpendicular from* **A** to **L.** Let Ψ *be the set of all points* **X** *in* $\mathbb{P}$ *such that* **XA** *is parallel to* **L** (*hence* **X** *cannot lie on* **AB**). *Then the set of all angle measures* $|\angle$**XAB**$|$, *taken over all* **X** *in* Ψ, *assumes a* ***minimum positive value*** Π**(A, B)** *which is always* ***strictly less than*** $90°$.■



In the drawing, the line **M** is given by **AC,** where $|\angle$**CAB**$|$ = Π**(A, B).** It follows that **M** is parallel to **L**, and the angle θ between **AB** and **M** (measured counterclockwise from **AB**) is as small as possible (***i.e.***, if the angle is smaller, then the line will meet **L**). Such a line in hyperbolic geometry is called a ***critically parallel*** (or ***asymptotically parallel***, or ***hyperparallel***) ***line***; in some books or papers such lines are simply called [hyperbolic] parallel lines. Similarly, the line **N** that forms the same angle θ between **AB** and itself but clockwise from **AB** will also be hyperparallel, but there can be no others. All other lines through **A** parallel to **L** form angles greater than θ with **AB**, and these are called ***ultraparallel*** (or ***disjointly parallel***) lines; this turns out to be the same as the class of line pairs which have common perpendiculars. Since there are an infinite number of possible angles between θ and 90 degrees, and each value will determine two lines through **A** that are ultraparallel to **L**, it follows that ***we have an infinite number of ultraparallel lines to* L *passing through* A.**

<u>**Notation.**</u> The number Π**(A, B)** is called the (Lobachevsky) ***critical angle*** or ***angle of parallelism*** for **L** and **A**, and it plays a fundamentally important role in hyperbolic geometry. As suggested above, a great deal of information about this number is contained in Moïse; for example, the value only depends upon $d = |$**AB**$|$, and the ***Bolyai – Lobachevsky Formula*** states that

$$\Pi(x) = 2\tan^{-1}(e^{-x})$$

where $x = d/k$ for some positive "curvature constant" we shall call k. The need to include the curvature constant k reflects the fact that similar triangles are always congruent in hyperbolic geometry, and in the 1824 letter from Gauss to Taurinus there are some comments about this constant:

> I can solve every problem in it [non – Euclidean geometry] with the exception of the determination of a constant, which cannot be designated ***a priori***. The greater one takes this constant, the nearer one comes to Euclidean geometry, and when it is chosen infinitely large the two coincide. ... If it [non – Euclidean geometry] were true, there must exist in space a linear magnitude, determined

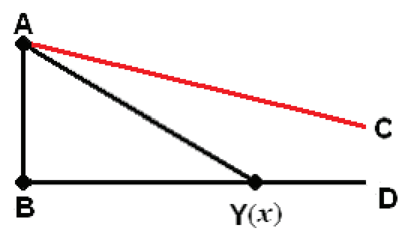
for itself (but unknown to us). ... If this non – Euclidean geometry were true, and it were possible to compare that constant with such magnitudes as we encounter in our measurements on the earth and in the heavens, it could then be determined ***a posteriori***. Consequently, in jest I have sometimes expressed the wish that the Euclidean geometry were not true, since then we would have ***a priori*** an absolute standard of measure.

<u>Further results on parallel lines in hyperbolic geometry.</u> One can now state the following more complete description of parallelism in hyperbolic geometry:

<u>Theorem 11.</u> *Let $\mathbb{P}$ be a hyperbolic plane. Given a line* **L** *in* $\mathbb{P}$*, and a point* **A** *not on* **L,** *there are exactly two lines through* **A** *which are critically parallel to* **L** *and infinitely many lines through* **A** *that are ultraparallel to* **L.**■

The previously stated asymptotic property of critical parallel lines is given by the next result:

<u>Theorem 12.</u> *Suppose we are given points* **L, A, B, X** *as above in the hyperbolic plane* $\mathbb{P}$ *such that* $|\angle \mathbf{CAB}| = \Pi(\mathbf{A}, \mathbf{B})$. *Let* **D** *be a point of* **L** *on the same side of* **AB** *as* **C.** *Given a positive real number* x, *let* $\mathbf{Y}(x)$ *be the unique point on* **(AD** *which satisfies* $|\mathbf{AY}(x)| = x$, *and let* $\sigma(x)$ *be the distance from* $\mathbf{Y}(x)$ *to the foot of the perpendicular to* **AC.** *Then the function* $\sigma(x)$ *is strictly decreasing and the limit of* $\sigma(x)$ *as* x *approaches* $+\infty$ *is equal to* 0.■



In the preceding result the function $\sigma(x)$ is defined for nonnegative values of x, and one can extend the definition of the function to negative values of x by first taking to be the unique point $\mathbf{Y}(x)$ on the ***opposite ray*** $[\mathbf{BD}^{\mathbf{OP}}$ such that $|\mathbf{AY}(x)| = x$, and then setting $\sigma(x)$ equal to the distance from $\mathbf{Y}(x)$ to the foot of the perpendicular to **AC.** We then have the following result:

<u>Complement 13.</u> *In the setting above, the function* $\sigma(x)$ *is strictly decreasing for all real values of* x, *and the limit of* $\sigma(x)$ *as* x *approaches* $-\infty$ *is equal to* $+\infty$.■

The preceding results imply that the graph of the function $\sigma(x)$ resembles the graph of the familiar exponential decay function $f(x) = e^{-x}$.

To complete the discussion, we shall state the corresponding result for ultraparallel lines.

<u>Proposition 14.</u> *Suppose we are given disjoint lines* **AC** *and* **BD** *in the hyperbolic plane* $\mathbb{P}$ *such that* **AB** *is perpendicular to both* **AC** *and* **BD,** *and assume that* **C** *and* **D** *lie on the same side of* **AB.** *Let* $\mathbf{Y}(x)$ *and* $\sigma(x)$ *be defined as in the preceding results. Then the function is an even function with* $\sigma(x) = \sigma(-x)$, *there* *is a minimum value at* $x = 0$, *the function is strictly increasing for positive values of* x *and strictly decreasing for negative values, and the limit of* $\sigma(x)$ *as* x *approaches* $+\infty$ *or* $-\infty$ *is equal to* $+\infty$.■

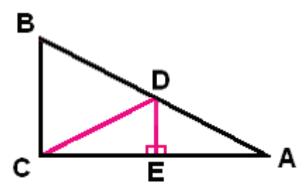
In these cases the graph of the function $\sigma(x)$ resembles the graphs of the familiar functions $f(x) = x^2$ and (more accurately) $f(x) = \frac{1}{2}(e^x + e^{-x})$.

Proofs of the statements about critical parallels and ultraparallels can be found in many books covering hyperbolic geometry. References for this and other material will be given at the beginning of the next section.

Appendix to Section 4: Solved exercises in neutral and hyperbolic geometry

Here are some further examples of problems similar to the exercises for these notes, followed by complete solutions.

PROBLEM 1. *Suppose that we are given a right triangle* $\triangle$**ABC** *in the hyperbolic plane* $\mathbb{P}$ *with a right angle at* **C**, *and let* **E** *denote the midpoint of* **[AB]**. *Prove that the line* **L** *perpendicular to* **AC** *through* **E** *contains a point* **D** *on* **(AB)** *and that* $|\mathbf{BD}|$ *is greater than* $|\mathbf{AD}| = |\mathbf{CD}|$.



SOLUTION. First of all, by Pasch's Theorem we know that the perpendicular bisector **L** either contains a point of **[BC]** or of **(AB)**. However, since **AC** is perpendicular to both **BC** and **L** we know that the first option cannot happen, and therefore the line **L** must contain some point **D** of **(AB)**. By **S.A.S.** we have $\triangle$**DEA** $\cong$ $\triangle$**DEC**, and therefore it follows that $|\mathbf{AD}| = |\mathbf{CD}|$. Furthermore, we have $|\angle\mathbf{DAE}| = |\angle\mathbf{DCE}|$. By the additivity property for angle measurements, we have

$$|\angle\mathbf{DAE}| + |\angle\mathbf{DCB}| = |\angle\mathbf{DCE}| + |\angle\mathbf{DCB}| = 90^\circ$$

and if we combine this with $\angle\mathbf{DAE} = \angle\mathbf{BAC}$, $\angle\mathbf{CBD} = \angle\mathbf{CBA}$, and the hyperbolic angle – sum property

$$|\angle\mathbf{BAC}| + |\angle\mathbf{CBD}| < 90^\circ$$

we see that $|\angle\mathbf{DBC}| < |\angle\mathbf{BCD}|$. Since the larger angle is opposite the longer side, it follows that $|\mathbf{BD}| > |\mathbf{CD}| = |\mathbf{AD}|$.■

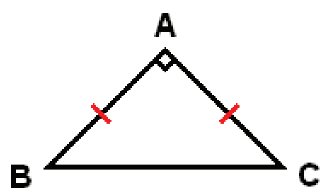
PROBLEM 2. *In the setting of the previous problem, determine whether* $|\angle\mathbf{BAC}|$ *is less than, equal to or greater than* $\frac{1}{2}|\angle\mathbf{BDC}|$.

SOLUTION. We have $|\angle ADC| = |\angle CDE| + |\angle EDA|$ because the midpoint **E** lies in the interior of $\angle ADC$, and since $\triangle DEA \cong \triangle DEC$ it also follows that $|\angle ADC| = 2|\angle EDA|$. By the supplement property for angle measures we have $|\angle BDC| + |\angle ADC| = 180°$. Therefore we also have $\frac{1}{2}|\angle BDC| + |\angle EDA| = 90°$. On the other hand, the hyperbolic angle – sum property implies that $|\angle BAC| + |\angle EDA| < 90°$. Therefore we have $|\angle BAC| + |\angle EDA| < \frac{1}{2}|\angle BDC| + |\angle EDA|$, and if we subtract the second term from each side of this inequality we conclude that $|\angle EDA| < \frac{1}{2}|\angle BDC|$.■

PROBLEM 3. *Suppose that we are given a right triangle* $\triangle ABC$ *in the neutral plane* $\mathbb{P}$ *with a right angle at* **C**, *and let* **F** *denote the midpoint of* **[AB]**. *Show that if* **F** *is equidistant from the vertices, then* $\mathbb{P}$ *is Euclidean.*

SOLUTION. If **F** is equidistant from the vertices, then **EF** is the perpendicular bisector of **[AC]**, and hence we must have **F = D.** However, by the first problem we know **D** is ***not*** equidistant from the vertices if the plane $\mathbb{P}$ is hyperbolic, and therefore $\mathbb{P}$ must be Euclidean.■

PROBLEM 4. *Suppose that we are given an isosceles triangle* $\triangle ABC$ *in the neutral plane* $\mathbb{P}$ *such that* $|AB| = |AC|$ *and* $|\angle BAC| > 60°$. *Prove that* $|BC| > |AC| = |AB|$.



NOTE: In hyperbolic geometry the same conclusion holds if |∠ BAC| = 60 (why?), but not so in Euclidean geometry (look at equilateral triangles).

Discussion. The drawing depicts an isosceles right triangle. As such, we know that its hypotenuse is longer than either of its legs, and this is in fact true in neutral geometry (the base angles, which have equal measure, must be acute, and the longer side is opposite the larger angle). The object of the exercise is to prove a more general result which is also true in neutral geometry.

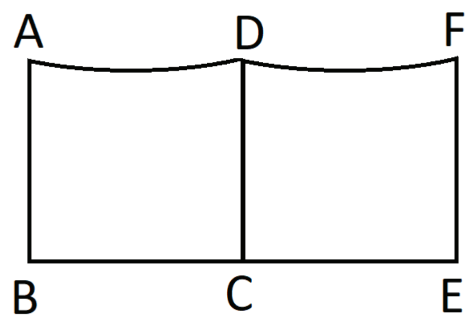
SOLUTION. By the Saccheri – Legendre Theorem we have

$$|\angle BAC| + |\angle ABC| + |\angle ACB| = |\angle BAC| + 2|\angle ABC| \leq 180°$$

and since $|\angle BAC| > 60°$ it follows that $2|\angle ABC| < 120°$, and therefore $|\angle ABC| < 60°$. Since the larger angle of a triangle is opposite the longer side, we must have $|BC| > |AB|$, and the final part of the conclusion follows because the right hand side is equal to $|AC|$.■

Note. In Euclidean geometry there is a companion result for isosceles triangles: *If* $|\angle BAC| < 60°$, *then* $|BC| < |AC| = |AB|$. — This is true because the angle – sum property in Euclidean geometry implies that $|\angle ABC| > 60°$ if $|\angle BAC| < 60°$. However, the companion result does not hold in hyperbolic geometry. In fact, under these conditions for a fixed value of $|\angle BAC|$ it is possible to construct triangles in a hyperbolic plane for which $|\angle ABC| = |\angle ACB|$ is arbitrarily small.

PROBLEM 5. *Assume that we are working in a hyperbolic plane* $\mathbb{P}$. *Suppose that we are given two side-by-side Saccheri quadrilaterals* **☐ABCD** *and* **☐DCEF** *such that* **B∗C∗E**, *each of* **AB, CD, EF** *is perpendicular to the line of* **B,C** and **E,** *the points* **A, D** *and* **F** *all lie on the same side of* **BC,** *and* $|AB| = |CD| = |EF|$. *Prove that* **A, D** *and* **F** *are noncollinear.*



In contrast, the three points are collinear in Euclidean planes.

SOLUTION. Since **AB ⊥ BC, CD ⊥ BC** and **EF ⊥ BE = BC,** each of the lines **AB, CD, EF** is parallel to the others. Now $B{*}C{*}E$ implies that **B** and **E** are on opposite sides of **CD,** and since all points of the parallel lines **AB** and **EF** lie on the same sides of **CD,** it follows that **A** and **F** lie on opposite sides of **CD.** It follows that **(AF)** meets **CD** at some point **X.**

Should be **B∗C∗E**

Assume that **A, D** and **F** are collinear. The points **X** and **D** are on both **AF** and **CD,** so it follows that **X = D.** Our assumption that $|AB| = |CD| = |EF|$ now implies that **A, B, E** and **F** (in that order) are the vertices of a Saccheri quadrilateral. Since the summit angles of a Saccheri quadrilateral have equal measures (see the exercises), we have $|\angle DAB| = |\angle DFE|$. But we also have Saccheri quadrilaterals **☐ABCD** and **☐DCEF,** and hence the equality of summit angles' measures implies that $|\angle ADC| = |\angle DAB| = |\angle DFE| = |\angle CDF|$. Combining this with **A∗D∗F** and the supplement postulate, we see that $|\angle ADC| = |\angle CDF| = 90°$. Therefore the Saccheri quadrilaterals **☐ABCD** and **☐DCEF** are both rectangles, and this contradicts our assumption that the neutral plane $\mathbb{P}$ is hyperbolic. The source of the contradiction is our assumption that **A, D** and **F** are collinear, and thus we conclude that **A, D** and **F** must be noncollinear. ■

Lecture 17 ends here.