Lecture 19
begins here

5 : Consistency and uniqueness of neutral geometries

> I have developed this geometry to my own satisfaction so that I can solve every problem that arises in it with the exception of the determination of a certain constant which cannot be determined ***a priori.***
>
> Gauss, previously cited letter to Taurinus, 1824

We have noted that the geometry and trigonometry of a hyperbolic plane were worked out completely in the early 19th century; more precisely, the formulas for rectangular and polar coordinate systems, trigonometry, and measurements and volumes are just as complete as they are for the Euclidean plane even though they are a usually great deal more complicated. A detailed treatment of this material is beyond the scope of this course. However, a great deal of appears in the book by Greenberg cited below; we shall also list several basic references for additional information about hyperbolic geometry.

H. S. M. Coxeter, ***Non – Euclidean Geometry*** (6th Ed.), Mathematical Association of America, Washington, DC, 1998.

R. Bonola, ***Non – Euclidean Geometry*** (Transl. by H. S. Carslaw). Dover, New York, 1955.

K. Borsuk and W. Szmielew, ***Foundations of Geometry*** (Rev. English Transl.). North Holland, Amsterdam (NL), 1960.

W. T. Fishback, ***Projective and Euclidean Geometry*** (2nd Ed.). Wiley, New York, 1969.

M. J. Greenberg, ***Euclidean and non – Euclidean geometries: Development and history*** (Fourth Ed.). *W. H. Freeman*, New York, NY, 2007.

P. Ryan, ***Euclidean and non – Euclidean geometry*: *An analytical approach.*** *Cambridge University Press*, Cambridge, U. K., and New York, NY, 1986.

H. E. Wolfe, ***Introduction to Non-Euclidean Geometry.*** Holt, New York, 1945.

http://www.msc.uky.edu/droyster/courses/spring08/math6118/

In this section we shall concentrate on a few topics that are closely related to previously discussed results in Euclidean geometry or are relevant to the remaining sections of this unit.

Uniqueness theorems for neutral geometries

We have abstractly defined a hyperbolic plane to be a system satisfying certain axioms. However, in mathematical writings one often sees references to **THE** hyperbolic plane as if there is only one of them, just as we talk about **THE** real number system or **THE** Euclidean plane. In all cases, the reason for this is that all such systems are characterized uniquely up to suitable notions of ***mathematical equivalence.***

Formally, this may be stated as follows:

Theorem 1. (Essential uniqueness of hyperbolic planes) *Assume that* $(\mathbb{P}, \mathcal{L}, d, \alpha)$ *and* $(\mathbb{P}', \mathcal{L}', d', \alpha')$ *are hyperbolic planes. Then there is a* $1-1$ *correspondence* **T** *from* $\mathbb{P}$ *to* $\mathbb{P}'$ *with the following properties:*

1. *If* **x** *and* **y** *are arbitrary distinct points of* $\mathbb{P}$, *then there is a positive constant* k *such that* **T** *multiplies the distance between them by* k; *in other words, we have* $k\,d(\mathbf{x}, \mathbf{y}) = d'(\mathbf{T}(\mathbf{x}), \mathbf{T}(\mathbf{y}))$.
2. *The function* **T** *sends collinear points (with respect to* $\mathcal{L}$*) to collinear points (with respect to* $\mathcal{L}'$*) and noncollinear points (with respect to* $\mathcal{L}$*) to noncollinear points (with respect to* $\mathcal{L}'$*).*
3. *If* **x, y, z** *are noncollinear points of* $\mathbb{P}$, *then* **T** *preserves the measurement of the angle they form; in other words, we have*

$$\alpha\angle\mathbf{xyz} = \alpha'\angle\mathbf{T(x)T(y)T(z)}.$$

The important point about the $1-1$ correspondence **T** is its compatibility with the data for the two hyperbolic planes. Using this mapping as a "codebook," it is possible to translate every true statement about one of the systems into a true statement about the other, and likewise it every false statement about one system translates into a statement which is also false for the other system.

In principle, this result for hyperbolic geometry was known to mathematicians such as Taurinus, Gauss, J. Bolyai and Lobachevsky, and it reflects their (essentially) complete description of the measurement formulas for non – Euclidean geometry and its associated trigonometry. Proofs of the uniqueness theorem are discussed further in Chapter 10 of Greenberg and Chapter VI (particularly Sections 30 and 31) of the previously cited book by Borsuk and Szmielew.■

If we define a neutral plane to be ***Euclidean*** if Playfair's Postulate is true, then there is a corresponding but slightly stronger uniqueness theorem for Euclidean planes:

Theorem 2. (Essential uniqueness of Euclidean planes) *Suppose that* $(\mathbb{P}, \mathcal{L}, d, \alpha)$ *and* $(\mathbb{P}', \mathcal{L}', d', \alpha')$ *are hyperbolic planes. Then there is a* $1-1$ *correspondence* **T** *from* $\mathbb{P}$ *to* $\mathbb{P}'$ *with the following properties:*

1. *If* **x** *and* **y** *are arbitrary distinct points of* $\mathbb{P}$, *then* **T** *preserves the distance between them; in other words, we have* $d(\mathbf{x}, \mathbf{y}) = d'(\mathbf{T}(\mathbf{x}), \mathbf{T}(\mathbf{y}))$.
2. *The function* **T** *sends collinear points (with respect to* $\mathcal{L}$*) to collinear points (with respect to* $\mathcal{L}'$*) and noncollinear points (with respect to* $\mathcal{L}$*) to noncollinear points (with respect to* $\mathcal{L}'$*).*
3. *If* **x, y, z** *are noncollinear points of* $\mathbb{P}$, *then* **T** *preserves the measurement of the angle they form; in other words, we have*

$$\alpha\angle\mathbf{xyz} = \alpha'\angle\mathbf{T(x)T(y)T(z)}.$$

Observe that a constant factor k does ***not*** appear in the statement of the Eucldean result. One way of explaining the difference is that there are similarity transformations in

Euclidean geometry with arbitrary positive rations of similitude, but in hyperbolic geometry every similarity transformation is an isometry (this reflects the conclusion of the hyperbolic **A.A.A.** Triangle Congruence Theorem; namely, similar triangles in a hyperbolic plane are automatically congruent).

The proof of the Euclidean uniqueness theorem reflects the standard method for introducing Cartesian coordinates into Euclidean geometry, and in principle the details are worked out in Chapter 17 of the previously cited book by Moïse (some material in Section 26.3 is also relevant).■

Euclidean approximations to hyperbolic geometry

For small enough regions on the surface of a sphere, ordinary experience and the explicit formulas of spherical trigonometry show that Euclidean plane geometry is a very accurate approximation to spherical geometry. The situation for hyperbolic geometry is entirely similar; if we restrict attention to sufficiently small regions, the formulas of hyperbolic trigonometry and geometry show that Euclidean geometry is an extremely accurate approximation and that the degree of accuracy increases as the size of the region becomes smaller. For example, since the angle defect of a hyperbolic triangle determines its area, it follows that the angle sum of a triangle is very close to **180** degrees for all triangles in a very small region of the hyperbolic plane. In both spherical and hyperbolic geometry, as the diameter of a region approaches zero, the formulas of spherical and hyperbolic geometry in the region converge to the standard formulas of Euclidean geometry.

Consistency models in mathematics

Although Gauss, J. Bolyai, Lobachevsky and others ***concluded*** that there was no way to prove Euclid's Fifth Postulate from the other assumptions, they did not actually ***prove*** this fact. Their results gave a virtually complete and apparently logically consistent description of hyperbolic geometry, but **something more was needed to *eliminate*, or at least isolate, *all doubts* that *someone might still succeed in finding a logical contradiction in the system***.

Mathematical statements that something cannot be found are frequently misunderstood, so we shall explain what is needed to show that a mathematical system is at least relatively free from logical contradictions. The discussion must begin on a somewhat negative note: Fundamental results of K. Gödel (1906 – 1978) imply that we can never be absolutely sure that any finite set of axioms for ordinary arithmetic (say, over the nonnegative integers) is totally free from logical contradictions. One far – reaching consequence is that ***there is also no way of showing that any infinite mathematical system is absolutely logically consistent***. The best we can expect is to show that such a system will be ***relatively logically consistent***; in other words, **if there is a logical contradiction in the system, then one can trace it back to a logical contradiction in our standard axioms for the nonnegative integers**. The following quotation due to André Weil (pronounced "VAY," 1906 – 1998) gives a whimsical reaction which reflects current mathematical thought:

God exists since mathematics is consistent, and
the Devil exists since we cannot prove it.

The standard way to prove relative logical consistency is to construct a ***model*** for the axioms. Such models are to be constructed using data based upon the standard number systems of mathematics (the nonnegative integers, the integers, the rational numbers, the real numbers or the complex numbers); the mathematical descriptions of these number systems show that all of them pass the relative consistency test described in the previous paragraph. If we can construct such a model, then one has the following ***proof for* RELATIVE *logical consistency***: Suppose that there is a logical contradiction in the underlying axiomatic system. Using the model, one can translate every statement about the model for the system into a statement about the mathematical number systems mentioned above, and thus the logical contradiction in the axiomatic system then yields a contradiction about these number systems. In other words, if there is a contradiction in the axioms, then there must also be a contradiction in the standard description of the standard number systems in mathematics.

If we consider the synthetic axioms for a Euclidean plane $(\mathbb{P}, \mathcal{L}, d, \alpha)$, the standard model is given by the so – called ***Cartesian coordinate plane***, in which the set $\mathbb{P}$ of points equal to $\mathbb{R}^2$, the family $\mathcal{L}$ of lines is the usual of family subsets defined by nontrivial linear equations in x and y, the distance d between two points is given by the usual Pythagorean formula, and the cosine of α is given by the standard formula involving inner products. In order to prove this is a model for the axioms, it is necessary to ***verify explicitly*** that ***all the axioms for Euclidean geometry***

(namely, the Incidence Axioms,
the Ruler and Plane Separation Postulates,
the Angle Measurement Postulates,
the Triangle Congruence Postulates
and the Proclus — Playfair Parallel Postulate)

are true for the given definitions of points, lines, distance and angle measure. Some steps in this process are fairly simple to complete, but others are long, difficult, and not particularly enlightening. It is frequently convenient to split the proof into two parts.

1. Replacement of the axiom system with an equivalent "reduced" one that requires fewer assumptions. (This can be long and difficult.)

2. Verification of the axioms in the "reduced" system.

We shall describe one relatively quick way of carrying out these steps for the Cartesian coordinate model of the synthetic axioms for Euclidean geometry. One particularly concise set of axioms for a Euclidean plane, consisting of only four statements, is given in the following classic paper:

G. D. Birkhoff, *A set of postulates for plane geometry (based on scale and protractors)*, ***Annals of Mathematics*** (2) **33** (1932), pp. 329 – 345.

A verification of Birkhoff's postulates for the Cartesian coordinate model is given explicitly in the following online document:

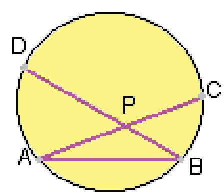
http://www.math.uiuc.edu/~gfrancis/M302/handouts/postulates.pdf

Alternate approaches to verifying the axioms for Euclidean geometry in the Cartesian model appear various sections of Moïse, mainly in Chapter 26.

Incidence axioms are not hard to verify, but the angle measure postulates are fairly difficult to check.

In view of the preceding discussion, it will follow that the Fifth Postulate is not provable from the other axioms if we can construct a model of a neutral plane $(\mathbb{P}, \mathcal{L}, d, \alpha)$ which does not satisfy Playfair's Postulate. The first such model was constructed by E. Beltrami (1835 – 1900) in his paper, *Saggio di interpretazoine della geometria non euclidea,* which appeared in 1868, with subsequent refinements due to F. Klein (1849 – 1925). This model is frequently called the ***Beltrami — Klein model.*** Much of the discussion below is adapted from the following online site:

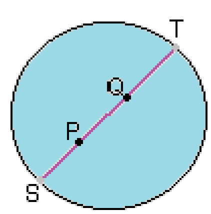
http://www.cut-the-knot.org/triangle/pythpar/Model.shtml



Asymptotic parallel lines are given by chords with one common endpoint.

The Beltrami – Klein model takes the interior of a circle as the set of ***points*** for a plane; recall that this region does not include points on the circle itself. The ***lines*** are given by open chords connecting points on the circle, with the endpoints excluded. It is not difficult to check that this system satisfies the basic incidence axioms, and the drawing above suggests an argument to show that Playfair's Postulate does not hold for points and lines in the Beltrami – Klein model. Specifically, in this picture the Beltrami – Klein lines (= open chords) **AC** and **BD** pass through point **P** and neither meets the **open** chord **AB** (with the endpoints removed).

Defining the distance and angle measurement for the Beltrami – Klein model is considerably more difficult; we shall only define the distance. Since hyperbolic geometry is unbounded, in order to realize it in a bounded region of $\mathbb{R}^2$, it is necessary to define distance so that the distance from one point to another goes to infinity if one is fixed and the other approaches the boundary circle.



Other advances in geometry led to the distance formula.

Given two points **P** and **Q** in the open disk, suppose that the Euclidean line joining them meets the circle at points **S** and **T.** Then the Beltrami – Klein distance between **P** and **Q** is defined by the following strange looking formula:

$$d_{BK}(P, Q) \quad = \quad \left|\log_e\left((|Q-S|\cdot|P-T|)/(|P-S|\cdot|Q-T|)\right)\right|$$

Here $|X - Y|$ denotes the ***Euclidean*** distance between **X** and **Y.** It is a routine exercise to check that if **Q** moves away from a fixed **P** staying on the same line, the Beltrami – Klein distance between the two points grows without bound. This curious property of the model sounds somewhat like a line from Shakespeare's play, **Hamlet** (Act II, Scene 2, line 234):

> I could be bounded in a nutshell, and count myself king of infinite space.

Compare to the earlier "method in madness" quote

Higher dimensions. There are analogs of the Beltrami – Klein model for hyperbolic n – space in every dimension $n \geq 3$.

Beltrami's model finally gave a definitive answer to questions about the role of Euclid's Fifth Postulate, showing that ***it is impossible to prove this postulate or an equivalent statement from the other usual sorts of axioms***. In many respects, this outcome is extremely ironic. Many of the early efforts to prove the Fifth Postulate were motivated by a belief that its inclusion was a logical shortcoming of the ***Elements***. For example, the title to Saccheri's work on the subject began with the words which translate to ***Euclid vindicated***, and the following quotation from a letter to J. Bolyai from his father Farkas (Wolfgang) Bolyai (1775 – 1856) expresses a similar view:

> I [also] thought … I was ready to … remove the flaw from geometry and return it purified to mankind.

In fact, as noted in the book by Moïse (see pages 158 – 159) the ***real*** vindication of Euclid took place with the construction of Beltrami's example, which showed that something like the Fifth Postulate is logically indispensable for the development of classical Euclidean geometry and indicates a very respectable level of insight on Euclid's part into the logical structure of deductive geometry.

Following the construction of the Beltrami – Klein model, several other models were also described, and a few will be described or referred to in the next section.

6 : Subsequent developments

In Section 2 we indicated how advances in mathematics during the 17th and 18th centuries provided an important background for the work which led to the emergence of non – Euclidean geometry. Mathematical knowledge increased at an even faster pace during the 19th century; one superficial way of seeing this is to compare the amount of space devoted to that period in Kline's ***Mathematical Thought from Ancient to Modern Times*** to the amount of space devoted to the entire period before 1800. In every area of the subject there were dramatic new discoveries, major breakthroughs in understanding, and substantially greater insight into logical justifications for the many advances of the previous three centuries. Within geometry, there were several major developments in addition to the emergence of non – Euclidean geometry. These include the systematic approach to curves and surfaces by the techniques of differential geometry, the establishment of projective geometry as a major branch of the subject, the explicit study of geometry in dimensions greater than 3, and the use of algebraic techniques to analyze geometrical constructions by unmarked straightedge and compass. In particular, during the 19th century mathematicians were finally able to show that the following three classical Greek problems cannot be solved using only a compass and an unmarked straightedge:

Advances in other branches of mathematics influenced geometry and vice versa

Impossibility shown by:

1. **Angle trisection.** *Given an arbitrary angle, find a second angle whose degree measure is one third the measure of the original angle.* **P. L. Wantzel, 1837**

2. **Circle squaring.** *Given an arbitrary circle, find a square whose enclosed area is equal to the area enclosed by the circle.* **F. von Lindemann, 1882**

3. **Cube doubling.** *Given an arbitrary cube, find a second cube whose volue is twice that of the original cube.* **P. L. Wantzel, 1837**

We shall see that such developments also turned out to have significant consequences for non – Euclidean geometry.

Riemann's approach to geometry

We have noted that the angle sum of a triangle in Euclidean geometry is always equal to $180°$, while the angle sum of a triangle in hyperbolic geometry is always less than $180°$. On the other hand, we know that the angle sum of a triangle in spherical geometry is always strictly greater than $180°$, and thus it is natural to ask if there is a unified setting which includes both neutral geometry and spherical geometry. The crucial steps to constructing such a framework were due to G. F. B. Riemann (1826 – 1866), and his viewpoint led to far – reaching changes in the mathematical, physical and philosophical answers to the question, "What is geometry?"

In 1854 Riemann gave an advanced mathematical lecture, *On the hypotheses which underlie geometry*, which had a near – revolutionary mathematical impact. The first step is to describe an n – dimensional object as one in which sufficiently small pieces have coordinate systems which correspond to coordinates in suitable regions of n – dimensional coordinate space; this concept is broad enough to include the geometries mentioned above and also reasonable surfaces in 3 – space. A second fundamental concept in the approach, often called a ***metric tensor*** or ***Riemannian metric***, provides a means for computing standard measurements including the angles at which nice curves intersect, the lengths of reasonably well – behaved curves, and the areas of surfaces. In two dimensions a Riemannian metric is an expression that can be written in local coordinates as a formal expression

Some parts of geoemtry do not fit into this approach, but a vast portion does

$$E(x,y)\,dx\,dx \;+\; 2F(x,y)\,dx\,dy \;+\; G(x,y)\,dy\,dy$$

where E, F and G all have (say) continuous second partial derivatives and satisfy the conditions $E > 0,\ G > 0,$ and $EG - F^2 > 0,$ The length of a parametrized curve $\gamma(t) = (x(t), y(t))$ is then given by integrating the square root of the function given below. The inequalities guarantee that this function is always positive.

$$s(t) \;=\; E(x(t),y(t))\,x'(t)^2 \;+\; 2F(x(t),y(t))\,x'(t)\,y'(t) \;+\; G(x(t),y(t))\,y'(t)^2$$

In Euclidean geometry $E = G = 1$ and $F = 0$

In Euclidean geometry the coefficient functions are $E = G = 1$ and $F = 0$. For the Beltrami – Klein model of hyperbolic geometry the expression is more complicated:

$$\frac{(1-y^2)\,dx\,dx \;+\; 2xy\,dx\,dy \;+\; (1-x^2)\,dy\,dy}{(1-(x^2+y^2))^2}$$

For "good" surfaces in 3 – space, the Riemannian metric yields the formula for arc length in multivariable calculus and elementary differential geometry.

For surfaces in 3-space, the metric is just the First Fundamental Form in differential geometry.

Riemann's legacy and elliptic geometry

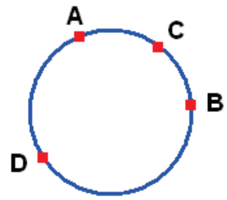
Riemann's setting provides a unified framework which encompasses spherical and neutral geometry. One key point was his questioning the standard model of a line, in which one can find pairs of points whose distance from each other is arbitrarily large, and it is summarized in the following quotation from his writings:

> We must distinguish between ***unboundedness*** and ***infinite extent*** ... The unboundedness of space possesses ... a greater empirical certainty than any external experience. But its infinite extent by no means follows from this.

With respect to our setting for neutral geometry, this means that the ***Standard Ruler Postulate*** should be replaced by a ***Circular Ruler Postulate*** which states that every line is in $1-1$ distance – preserving correspondence with a standard circle of some fixed positive radius (*i.e.*, the radius a is the same for every line); in analogy with our earlier discussion of hyperbolic geometry, the <u>square</u> a^2 of this radius can be viewed as a "curvature constant." If we adopt such a Circular Ruler Postulate, then we must also modify the entire discussion of order and separation. One easy way to see this is to observe that there is no reasonable notion of betweenness for three points on a circle. However, there is a decent substitute, for if we are given four points **A, B, C, D** on a circle then there is an obvious concept of ***separation*** for these points.

In calculus the curvature of a circle with radius a is $1/a^2$

Specifically, we can say that **A** ***and*** **B** ***separate*** **C** ***and*** **D** if each of the two arcs determined by **A** and **B** contains exactly one of the points **C** and **D.**



In the drawing above, the points **A** *and* **B** *separate* **C** *and* **D.**

We shall not attempt to make the concept of separation precise here, but the previously cited books by Fishback and Coxeter contain further information. Here are two additional references which discuss further topics in elliptic geometry:

H.S.M. Coxeter, ***The Real Projective Plane*** (3rd Ed.), Springer – Verlag, New York, 1992. ISBN: 0–387–97890–9.

http://eom.springer.de/R/r081890.htm

There is still one fundamental issue that requires attention. In neutral geometry there is a unique line containing two points, but the analogous statement in spherical geometry is not necessarily valid because there are infinitely many great circles joining a pair of antipodal points. An idea due to F. Klein provides the usual way of avoiding this problem: Instead of considering the geometry of the sphere, one considers a ***reduced geometry*** whose points are ***antipodal pairs of points on the sphere.*** Such a construction was not really new, for it is essentially the underlying space in real projective plane geometry (see **http://math.ucr.edu/~res/progeom/pg-all.pdf**)

This was partly motivated by the theory of perspective drawing from the late Middle Ages, with sustained attention beginning just before 1800

Klein's motivation for the name ***hyperbolic geometry*** suggests the name ***elliptic geometry*** for the system that one obtains from spherical geometry by identifying pairs of antipodal points as above; sometimes elliptic or spherical geometry is called ***Riemann*** or ***Riemannian geometry***, but in mathematics and physics these terms normally refer to far more general constructions and thus ***almost any other terms would be preferable***. There is a corresponding name of ***parabolic geometry*** for Euclidean geometry, but this name has never been popular with mathematicians and is so rarely used in modern mathematical writings that it should be viewed as obsolete.

Riemann's characterization of classical geometries

The value of non – Euclidean geometry lies in its ability to liberate us from preconceived ideas in preparation for the time when exploration of physical laws might demand some geometry other than Euclidean.

G. F. B. Riemann

Riemann's unified approach to spherical and neutral geometry is merely part of a far more general approach of geometry. The emergence of non – Euclidean geometry had suggested to Gauss and others that there was more than one "logically permissible" way of looking a space, depending upon which geometric properties one was willing to accept or do without. Riemann's viewpoint abandoned the idea that geometry involved absolute statements about space itself, replacing this with a premise that geometry

Compare the J. J. Thomson quote at the beginning of these notes.

involves the study of theories of space. In Riemann's approach, one has infinitely many possible theoretical options for describing space.

Even if Euclidean, hyperbolic and elliptic geometry represent only three of many possible theories of space, it is still clear that they represent three especially good theories. Therefore one of Riemann's central aims was to give a criterion for distinguishing these three from the unending list of possibilities. Within his framework, the three classical geometries are characterized by two special properties:

1. The existence of many different geometrical figures isometric to a given one.
2. A real number which is describable as a ***curvature constant***.

For Euclidean geometry the curvature constant is zero, while for hyperbolic geometry it is negative and for elliptic geometry it is positive; in the last two cases, the exact value depends upon the unit of linear measurement one adopts. In elliptic geometry, the square root of the curvature constant is the reciprocal of the radius for the corresponding sphere; the negativity of the curvature constant for hyperbolic geometry is related to Lambert's view of the latter in terms of "a sphere of imaginary radius" (*i.e.*, the square of the radius is ***negative***).

Additional models for hyperbolic geometry

Most of the ties between hyperbolic geometry and other topics in mathematics involve mathematical models for the hyperbolic plane (and spaces of higher dimensions) which are different from the Beltrami – Klein models described in the preceding section. There are three particularly important examples. One model (the ***Lorentzian model***) is discussed at length in Chapter 7 of Ryan, and two other basic models are named after H. Poincaré (pronounced *pwan — ca — RAY*). We shall only consider a few of properties of the Poincaré models in these notes. Further information can be found at the following online sites:

http://www.geom.uiuc.edu/docs/forum/hype/model.html

http://www.mi.sanu.ac.yu/vismath/sazdanovic/hyperbolicgeometry/hypge.htm

http://math.fullerton.edu/mathews/c2003/poincaredisk/PoincareDiskBib/Links/PoincareDiskBib_lnk_1.html

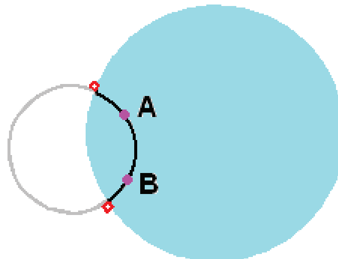
http://mathworld.wolfram.com/PoincareHyperbolicDisk.html

http://www.geom.uiuc.edu/~crobles/hyperbolic/hypr/modl/

Probably the most important and widely used model for hyperbolic geometry is the ***Poincaré disk model***. In the $2-$dimensional case, one starts with the points which lie in the interior of a circle (*i.e.*, in an open disk) as in the Beltrami – Klein model, but the definitions of lines, distances and angle measures are different. The lines in this model are given by two types of subsets.

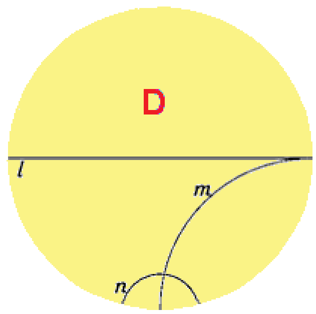
1. Open "diameter" segments with endpoints on the boundary circle.
2. Open circular arcs whose endpoints lie on the boundary circle and meet the boundary circle ***orthogonally*** (*i.e.*, at each endpoint, the tangent to the boundary circle is perpendicular to the tangent for the circle containing the arc).

An illustration of the second type of "line" is given below.



*The arc containing **A** and **B** meets the boundary of the disk at right angles.*

The drawing below illustrates several lines in the Poincaré disk model.



As in the Beltrami-Klein model, asymptotcally parallel lines are those with one end point in common

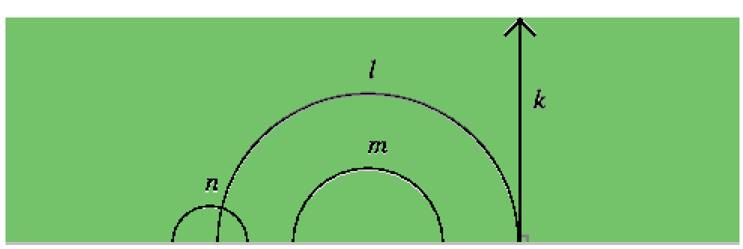
The Poincaré disk model distance between two points is given by a formula which resembles the comparable identity for the Beltrami – Klein model, and it is given in the first online reference in the list of online sites at the beginning of this section. On the other hand, one fundamentally important feature of the Poincaré disk model is that ***its angle measurement is exactly the same as the Euclidean angle between the two intersecting curves*** (*i.e.*, given by the usual angle between their tangents); such angle measure preserving models are said to be ***conformal***. In contrast, both the distance and the angle measurement in the Beltrami – Klein model are different from their Euclidean counterparts.

The second Poincaré model in two dimensions is the ***Poincaré half – plane model***, and its points are given by the points in the upper half plane of $\mathbb{R}^2$; in other words, the points are all ordered pairs (x, y) such that $y > 0$. The lines in this model are once again given by two types of subsets.

(1) Vertical open rays whose endpoints lie on the x – axis.

(2) Open semicircular arcs whose endpoints lie on the x – axis.

The drawing below illustrates several lines in the Poincaré half – plane model.



For this model the asymptotically parallel lines are also curves with one endpoint in common

The Poincaré half – plane model distance between two points is given by a formula in the first reference in the list of online sites at the beginning of this section. As in the preceding case, one fundamentally important feature of the Poincaré half – plane model is that its angle measurement is exactly the same as the Euclidean angle between two intersecting curves (*i.e.*, given by the usual angle between their tangents).

For at least a century, the two Poincaré models have been the most widely used ones for studying hyperbolic geometry.

One reason for this involves important relations between the models and the subject of ***complex variables***. Further information can be found in many textbooks on that subject. The following textbook is a specific example:

> S. Lang, ***Complex Analysis*** (4th Ed., corrected 3rd printing). Springer – Verlag, New York, 2003. ISBN: 0–387–98592–1.

Poincaré trivia. The renowned mathematician and physicist (Jules) Henri Poincaré (1854 – 1912) was a cousin of Raymond Poincaré (1860 – 1934), who was President of France from 1913 to 1920 and Prime Minister of France for three terms (1912 – 1913, 1922 – 1924, and 1926 – 1929).

Appendix to Section 6: Euclidean models of hyperbolic geometry

This material may be skipped without loss of continuity.

One important property of hyperbolic 3 – space is that it contains a surface called a ***horosphere*** which is isometric to the Euclidean plane. The hyperbolic plane and its properties might have been discovered sooner if mathematicians had previously found a surface in Euclidean space which were isometric to the hyperbolic plane.

It would be very satisfying if we could give a nice model for the hyperbolic plane in Euclidean 3 – space for which the distance is something more familiar (*i.e.*, the hyperbolic distance between two points is the length of the shortest curve in the model joining these points), but unfortunately this is not possible. The first result to show that no reasonably nice and simple model can exist was obtained by D. Hilbert (1862 – 1943) in 1901, and it was sharpened by N. V. Efimov (1910 – 1982) in the 1950s. One reference for Hilbert's Theorem is Section $5 - 11$ in the following book:

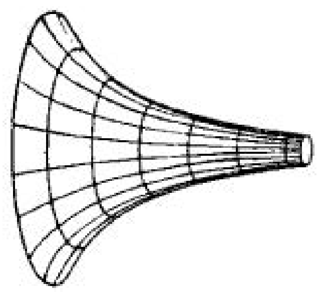
> M. Do Carmo, ***Differential Geometry of Curves and Surfaces***, Prentice – Hall, Upper Saddle River, NJ, 1976.

In contrast, during 1950s N. H. Kuiper (1920 – 1994) proved a general result which shows that the hyperbolic plane can be realized in Euclidean 3 – space with the "right" distance, but the proof is more of a pure existence result than a method for finding an explicit example, and in any case the results of Hilbert and Efimov show that any such example could not be described very simply. Kuiper's result elaborates upon some fundamental results of J. F. Nash (1928 – 2015); another an extremely important general result of Nash implies that the hyperbolic plane can be realized nicely in Euclidean n – space if n is sufficiently large; it is known that one can take $n = 6$, but apparently there are open questions about the existence of such realizations if $n = 5$ or 4. Here are two references for the realizability of the hyperbolic plane in Euclidean 6 – space; the first is the original paper on the subject, and the second contains a fairly explicit construction of a nice model near the end of the file.

D. Blanuša, *Über die Einbettung hyperbolischer Räume in euklidische Räume.* Monatshefte für Mathematik **59** (1955), 217 – 229.

http://www.math.niu.edu/~rusin/known-math/99/embed_hyper

In yet another — and more elementary — direction, **it is not particularly difficult to represent SMALL PIECES of the hyperbolic plane nicely in Euclidean 3 – space**. In particular, this can be done using a special surface of revolution known as a ***pseudosphere***. Further information on this surface can be found in many differential geometry books.



The pseudosphere is a surface of revolution for a curve called the ***tractrix***; this surface extends infinitely to the right, becoming increasingly narrow as one moves in that direction such that the radii of the circles go to zero at infinity. If one removes the left hand boundary circle and the original tractrix curve, the remaining part of the surface is isometric to a region in the hyperbolic plane.

Biographical footnote. (*This is basically nonmathematical information.*) The extraordinary life of John Nash received widespread public attention in the biography, ***A Beautiful Mind***, by S. Nasar (1974 –), and the semi – fictional interpretation of her book in an Academy Award winning film of the same name. During the 1950s Nash proved several monumental results in geometry including the embedding theorem cited above. However, in nonmathematical circles he is better known for his earlier work on game theory, for which he shared the 1994 Nobel Prize in Economics with J. Harsányi (1920 – 2000) and R. Selten (1930 – 2016); an ironic apsect of this is noted in the footnote at the bottom of page 565 in the book by Greenberg.

A more detailed account of the historical ties between geometry and physics is beyond the scope of these notes, but a fairly readable and detailed account of the history into the early 20th century is contained in the following book:

C. Lanczos, ***Space through the Ages: The evolutions of geometric ideas from Pythagoras to Hilbert and Einstein,*** Academic Press, New York, 1970. ISBN: 0–124–35850–0.

7 : The impact of hyperbolic geometry

> Hyperbolic geometry, which was considered a dormant subject … [around the middle of the 20th century], has turned out to have extraordinary applications to other branches of mathematics.
>
> Greenberg, p. 382

It seems appropriate to conclude these notes on non – Euclidean geometry with a brief discussion of the role it plays in present day mathematics. Questions of this sort arise naturally, and In particular ***one might ask whether objects like the hyperbolic plane are basically formal oddities or if they are important for reasons beyond just showing the logical independence of the Fifth Postulate***. In fact, **hyperbolic geometry turns out to play significant roles in several contexts of independent interest. Some of these date back to the 19th century, but others were first discovered during the last few decades of the 20th century**.

Numerous properties of hyperbolic n – spaces play fundamental roles in many aspects of that subject, including some that have seen a great deal of progress over the past four decades. Three books covering many of these advances are discussed in a relatively recent book review by B. Kleiner [Bull. Amer. Math. Soc. (2) **39** (2002), 273 – 279.] We shall only discuss a few topics that are relatively easy to explain.

Philosophical and practical consequences

In the introduction to these notes, we noted that the emergence of non – Euclidean geometry had a strong impact on the philosophy and foundations of mathematics. In the next few paragraphs we shall describe this impact in more detail.

Background. At the beginning of the 19th century, Euclidean geometry was viewed as a reliable foundation for mathematics. Its importance for geometry is evident, but it was also important for algebra; in particular, very large portions of the Elements involve the use of geometrical methods to study irrational numbers. The reasons for this heavy emphasis on geometry are described in the following passage from M. Kline's ***Mathematics and the Physical World*** (Corrected reprint of the 1959 Ed., Dover, New York, 1981.):

> As of 1800, mathematics rested upon two foundations: The number system and Euclidean geometry. … Mathematicians would have emphasized the latter because many facts about the number system, and about irrational numbers especially, were not logically established nor clearly understood. Indeed, those properties of the number system that were universally accepted were still proved by resorting to geometric arguments, much as the Greeks had done 2500 [probably more like 2100] years earlier. Hence, one could say that Euclidean geometry was the most solidly constructed branch of mathematics.

These ideas are explicit in following two quotations from the writings of Isaac Barrow (1630 – 1677), who is best known mathematically for his contributions to calculus and for teaching Isaac Newton:

> Geometry is the basic mathematical science, for it includes arithmetic, and mathematical numbers are simply the signs of geometrical magnitude.

> Geometry is certain [contrary to the infinitesimal calculus] because of the clarity of its concepts, its unambiguous definitions, our intuitive assurance of the universal truth of its common notions, the clear possibility and easy imaginability of its postulates, the small number of its axioms ...

This viewpoint is also implicit in Isaac Newton's monumental work ***Principia***, which uses Euclidean geometry as its logical foundation.

At the beginning of these notes, we included a quotation from Kant reflecting his view of Euclidean geometry as description of ***a priori*** truths, just like the fundamental rules for arithmetic. His viewpoint on such ***a priori*** truths is reflected in the following passage from ***The Story of Philosophy*** (Pocket Books, Simon and Schuster, New York, 1991.) by W. Durant (1885 – 1981):

> We may believe that the sun will "rise" in the west tomorrow, or that … fire will not burn [a] stick, but we cannot for the life of us believe that two times two will ever make anything else than four. Such truths are true before experience … they are absolute and necessary; it is inconceivable that they should ever become untrue. … These truths derive their necessary character from the inherent structure of our minds, from the natural and inevitable manner in which our minds must operate.

As suggested by the quotation from Gauss at the beginning of this unit, the discovery of non – Euclidean geometry and the logical independence of the Fifth Postulate provided compelling evidence that the standard axioms for Euclidean geometry are not ***a priori*** truths.

The preceding developments had several implications. One was a need to give a new description of geometry, and this was done along the lines indicated in the following quotation from Kline's ***Mathematics in Western Culture*** (Oxford University Press, New York, 1964):

> A [geometric] mathematical space now takes on the nature of a scientific theory. … The creation of the new geometries … forced recognition of the fact that there could be an "**if**" about mathematical systems. ***If*** the axioms of Euclidean geometry are truths about the physical world then the theorems are. But … we cannot decide on ***a priori*** grounds that the axioms of Euclid, or of any other geometry, are [empirical] truths [about the physical world].

A second implication was the need to replace the role of Euclidean geometry as a foundation for mathematics by something else; actually, the discoveries related to the Fifth Postulate were just one of many factors which forced mathematicians to look more carefully at the foundations of the subject during the 19^{th} century and to find solid logical justifications for the spectacular advances the subject had made during the preceding three centuries. By the end of the 19^{th} century the modern approach to the foundations of mathematics had essentially been outlined with **(1)** the development of set theory, **(2)** the simple axiomatic characterization of the positive integers due to G. Peano (1858 – 1932), and **(3)** the formal construction of the real number system in terms of the rational numbers and characterization of the real numbers due to R. Dedekind (1831 – 1916). Each of these stands as a major achievement for separate reasons. In particular, Peano's axioms effectively answered questions by philosophers such as John Stuart Mill (1806 – 1873) about the ***a priori*** nature of arithmetic, and Dedekind's work finally resolved basic questions about irrational numbers which had been unanswered ever since the Pythagoreans discovered that the square root of 2 is irrational.

Some further advances. Riemann's insights opened the door to many new directions in geometrical research. For our purposes, it will suffice to say that the work led to more refined characterizations of the classical non – Euclidean geometries, particularly in the work of H. von Helmholtz (1821 – 1894) and S. Lie (1842 – 1899).

Practicality and convenience. Of course, the study of mathematics has a huge practical component, so the practical impact of non – Euclidean geometries should also be considered. We have already noted that Euclidean geometry is a good approximation to either spherical or hyperbolic geometry if one restricts attention to a fairly small region. Since the formulas of Euclidean geometry are much simpler than those of the other geometries, for practical purposes it is generally more convenient to work inside Euclidean geometry unless the region under consideration is fairly large. The relative convenience of Euclidean geometry provides one answer to the issue raised in Poincaré's statement which we quoted at the beginning of these notes.

Geometry and modern physics

> The value of non – Euclidean geometry lies in its ability to liberate us from preconceived ideas in preparation for the time when exploration of physical laws might demand some geometry other than Euclidean.
>
> G. F. B. Riemann

Although the emergence of non – Euclidean geometry raised immediate questions whether the physical universe satisfies the axioms of geometry, the real impact of these developments on physics did not begin for some time. We have noted that Euclidean geometry provides an excellent approximation to hyperbolic and elliptic geometry in small regions, and until the end of the 19th century experimental observations and classical physics were consistent with the mathematics of Euclidean geometry. However, near the end of the century physicists found that classical physics did not provide adequate explanations for some key experimental observations, and this led physicists to consider new mathematical models which would conform more closely to experimental results. Efforts by H. Lorentz (1853 – 1928) and G. FitzGerald (1851 – 1901) to explain the results of one important experiment led to a generalization of Riemann's geometric structures (***Lorentzian geometry***) that was a precursor to the Theory of Special Relativity introduced by A. Einstein (1879 – 1955) in 1905. Further extensions of Riemann's ideas led to the mathematical theory of space – time that underlies General Relativity Theory. Many other systems that can be called "theories of space" also appear in many contexts of 20th century (and present day) physics.

We conclude this discussion by mentioning two frequently misstated or misunderstood points about Einstein's work and its relation to non – Euclidean geometry.

Is the geometry of relativity theory a non – Euclidean geometry? **The answer to this question depends upon how one defines non – Euclidean geometry.** One basic point in relativity theory is that the presence of mass warps or curves the structure of space – time. In Euclidean geometry there is no curvature whatsoever, and thus it is clear that the geometry of space – time cannot be Euclidean. Furthermore, since the distribution of mass in the universe varies from place (and time) to place (and time), the curvature of space – time is also variable. In the classical non – Euclidean geometries (hyperbolic and elliptic), the curvature is nonzero but the same at all points. This means that ***the relativistic geometry of space – time is neither Euclidean, hyperbolic, nor elliptic.*** Therefore the answer to the question at the beginning of this paragraph depends upon whether non – Euclidean means *anything that is not Euclidean* (in which case the answer is **YES**) or means *only the classical examples* of hyperbolic and elliptic geometry (in which case the answer is **NO**).

What was Einstein's role in developing the mathematics of relativistic geometry? The basic mathematical framework for relativistic geometry had been previously created by others, and **Einstein's fundamental insight was to see that this framework was useful for formulating certain fundamental laws of physics**. Einstein's chief ***mathematical*** contribution was a geometrical formula relating the curvature properties of space – time to the distribution of matter in the universe (the so – called ***Einstein tensor equation***). Here is a reference for a precise statement of this equation:

http://en.wikipedia.org/wiki/Einstein_field_equation

A more detailed account of the historical ties between geometry and physics is beyond the scope of these notes, but a fairly readable and detailed account of the history into the early 20th century is contained in the following book:

> C. Lanczos, ***Space through the Ages: The evolutions of geometric ideas from Pythagoras to Hilbert and Einstein,*** Academic Press, New York, 1970. ISBN: 0–124–35850–0.

Regular tessellations

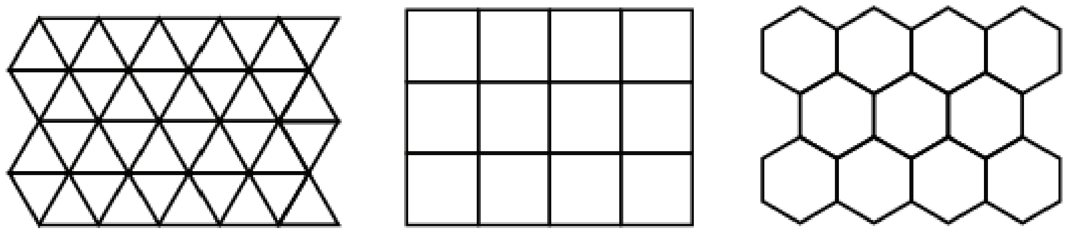
> The investigation of the symmetries of a given mathematical structure has always yielded the most powerful results.
>
> E. Artin (1898 – 1962)
>
> For some minutes Alice stood without speaking, looking out in all directions over the country ... "I declare it's marked out just like a large chessboard ... all over the world — if this is the world at all."
>
> Lewis Carroll (C. L. Dodgson, 1832 – 1898), ***Through the Looking Glass***
>
> A mathematician is … a maker of patterns.
>
> G. H. Hardy (1877 – 1947), ***A Mathematician's Apology***

Although a precise and comprehensive description of hyperbolic geometry's place in modern mathematics is beyond the scope of these notes, we shall describe one geometric manifestation of its role. However, before doing so we shall summarize the corresponding results for Euclidean and spherical geometry.

The planar case. A ***regular tessellation*** (or tiling) of the Euclidean plane is a decomposition of the plane into closed regions bounded by regular convex polygons such that the following hold:

1. All the bounding polygons have the same number of sides.
2. If the intersection of two distinct regions is nonempty, then it is a common side or vertex of the bounding polygons.

There are three obvious ways to construct such regular tilings of the Euclidean plane. If the regular polygons are squares, then one example corresponds to covering a flat surface by square tiles that do not overlap each other, and if the regular polygons are hexagons, then another example corresponds to the familiar honeycomb configuration of hexagons. A third example this type is the covering of a flat surface by tiles that are equilateral triangles. All of these are illustrated below.



Greek mathematicians (probably as early as the Pythagoreans) realized that the preceding examples were the only ones (up to similarity).

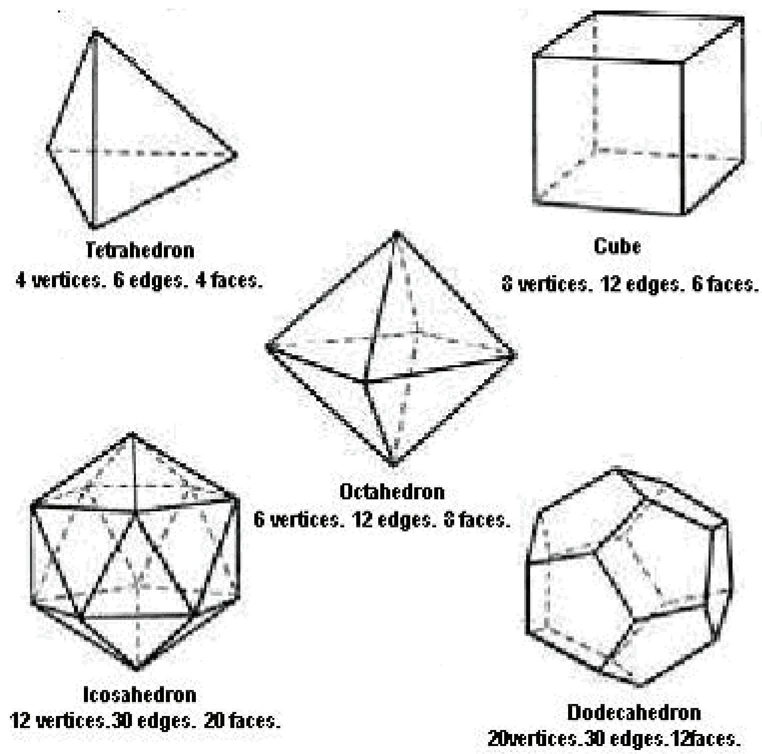
<u>The spherical case.</u> On the surface of the sphere, the regular tessellations correspond to ***regular polyhedra*** whose vertices lie on the sphere. More precisely, the vertices of the regular tessellation for the sphere are just the vertices of the regular polyhedron, its edges are the great circles joining two vertices (one for each pair of vertices which lie on a common edge of the polyhedron), and its faces are the regions bounded by the spherical polygons bounded by appropriate edges.

In view of the preceding discussion, the description of regular tessellations for the sphere reduces to the description of the possible types of regular polyhedra. The two simplest examples of regular polyhedra are a cube and a triangular pyramid such that each face is an equilateral triangle. Each of these illustrates the fundamental properties that all regular solids should have.

1. Every $2-$dimensional face should be a regular $n-$gon for some fixed value of $n \geq 3$.
2. Every $1-$dimensional edge should lie on exactly two faces.
3. Every vertex should lie on r distinct faces for some fixed value of $r \geq 3$.
4. No three vertices are collinear.
5. Given a face **F**, all vertices that are not on **F** lie on the same side of the plane containing **F**.

Regular polygons beyond triangles and squares were known in prehistoric times, and in fact archaeologists have also discovered early examples of stones carved and marked to represent several (in fact, most and maybe all) $3-$dimensional regular polygons. One major achievement of Greek mathematics (which appears at the end of Euclid's ***<u>Elements</u>***) was the proof that ***<u>there are exactly five distinct types of regular polyhedra</u>***, and they are listed and illustrated below:

Type of polyhedron	No. of vertices	No. of edges	No. of faces
tetrahedron	4	6	2
cube	8	12	6
octahedron	6	12	8
icosahedron	12	30	20
dodecahedron	20	30	12



(***Source:*** http://www.goldenmeangauge.co.uk/platonic.htm)

Examples in the hyperbolic plane

The situation in the hyperbolic plane is entirely different. One important reason is given by the following result, which is also mentioned on page 176 of the book by Ryan.

Theorem 2. *Let n be an integer greater than 2, and let θ be a positive number less than $180(n-2)/n$. Then there is a regular hyperbolic $n-$gon such that all the sides have equal length and the measures of all the vertex angles are equal to θ.*

In particular, if n is greater than 4 and $\theta = 90$, then one might expect that we can form a regular tessellation of the hyperbolic plane with regular $n-$gons such that four meet at each vertex. In fact, this is possible. This is a special case of the following general result:

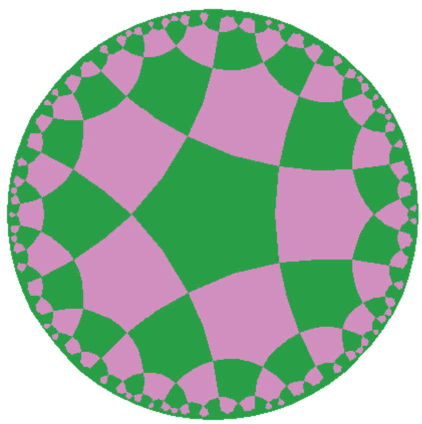
Theorem 3. *Suppose that* $m, n \geq 3$ *are integers such that*

$$\frac{1}{m} + \frac{1}{n} < \frac{1}{2}.$$

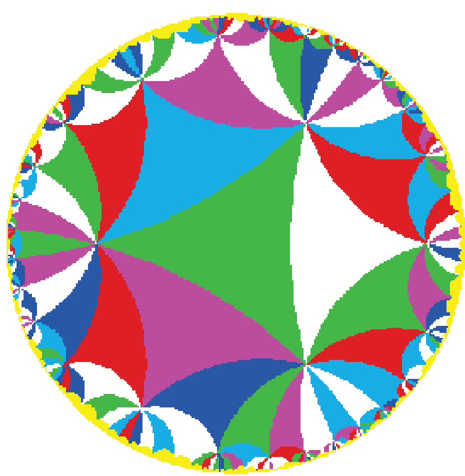
Then there is a regular tessellation of the hyperbolic plane into solid regular $n-$ *gons with* m *distinct polygons meeting at each vertex. Conversely, if there is a regular tessellation of the hyperbolic plane into solid regular* $n-$ *gons with* m *distinct polygons meeting at each vertex, then the displayed inequality holds.*

There are several ways to prove this theorem. In particular, algebraic results of W. F. von Dyck (1856 – 1934) give an approach which is related to the viewpoint of Ryan's book.

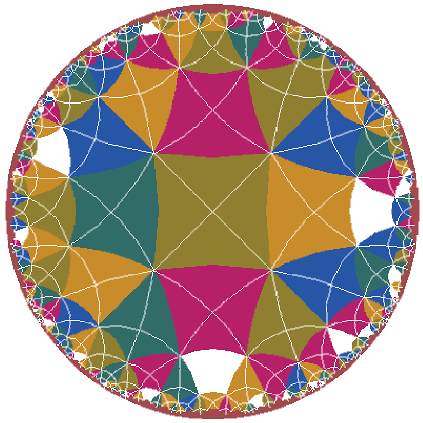
Since there are infinitely many pairs of positive integers m and n satisfying these conditions, it follows that ***there are infinitely many distinct regular tessellations of the hyperbolic plane.*** The type of such a tessellation is generally denoted by the ordered pair (n, m). A few illustrations are given below.



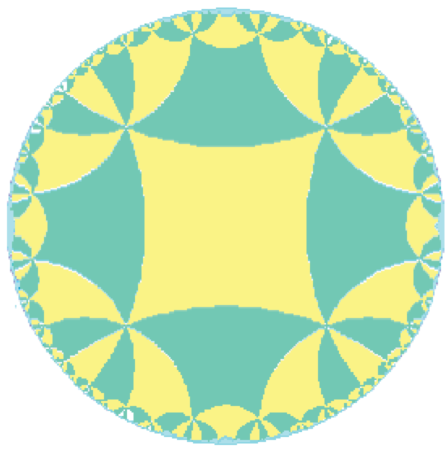
(5, 4)



(3, 12)



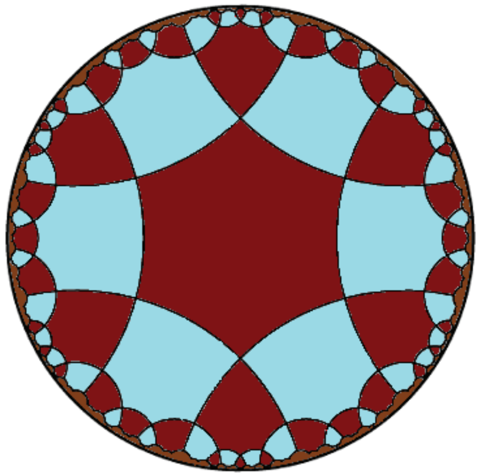
(4, 8)



(4, 6)

Regular tessellations of the hyperbolic plane also appear in some of the artwork created by M. C. Escher (1898 – 1972). For example, the angels and devils in the picture ***Circle Limit*** **IV** fit together to form a tessellation by regular hexagons with right angles at every vertex (type **(6, 4)** in our notation). This can be seen from the illustrations below:

The name is pronounced ESS-kher where kh is pronounced as in Bach



This is Escher's *Circle Limit IV*

A unified perspective on tessellations

There is an interesting relationship between Theorem $\mathbf{3}$ and the results for regular tessellations of the sphere and the Euclidean plane. In the Euclidean plane, there is a regular tessellation into solid regular $n-$gons with m distinct polygons meeting at each vertex if and only if

$$\frac{1}{m} + \frac{1}{n} = \frac{1}{2}.$$

because this equation holds if and only if (n, m) is equal to one of the three ordered pairs $(3, 6)$, $(4, 4)$ or $(6, 3)$. Similarly, on the sphere there is a regular tessellation of the hyperbolic plane into solid regular spherical $n-$gons with m distinct polygons meeting at each vertex if and only if

$$\frac{1}{m} + \frac{1}{n} > \frac{1}{2}$$

because this equation holds if and only if (n, m) is equal to one of the five ordered pairs $(3, 3)$, $(3, 4)$, $(3, 3)$, $(3, 5)$ or $(5, 3)$. If we combine these observations with Theorem **3**, we obtain the following unified conclusion:

> For each ordered pair (n, m) such that $m, n \geq 3$, there is a regular tessellation of either the ***Euclidean plane***, the ***hyperbolic plane*** or the ***sphere*** into solid regular $n-$ gons with m distinct polygons meeting at each vertex, and the specific type of plane supporting such a configuration is given by comparing ½ to the previously described sum of reciprocals:
>
> $$\frac{1}{m} + \frac{1}{n}$$
>
> In particular, the relevant geometry will be ***spherical*** if this sum is ***greater than*** ½, it will be ***Euclidean*** if this sum is ***equal to*** ½, and it will be ***hyperbolic*** if this sum is ***less than*** ½.

Final remarks

It seems appropriate to end these notes with the following quotation from page 105 of the book by Greenberg:

> Let us not forget that no serious [sustained] work toward constructing new axioms for Euclidean geometry had been done until the discovery of non – Euclidean geometry shocked mathematicians into reexamining the foundations of the former **[*Comment*:** Other considerations also played important roles in forcing a review of the foundations for classical geometry**]**. We have the paradox of non – Euclidean geometry helping us to better understand Euclidean geometry!