Readings for Unit I from Ryan
(Topics from linear algebra)

I.0 : Background

Suggested readings.

Ryan : pp. 193 – 202

> ***General convention.*** In most cases, the background readings for a section of the course will begin with a subheading on the initial page and will end just before a subheading on the final page.

Comments.

This is Appendix **D** in Ryan's book, and it is an extremely brief and rapid summary of basic linear algebra. Some first courses in linear algebra do not cover some of the topics discussed in these pages of Ryan, most notably multilinear functions (pp. 199 – 200) and eigenvectors and eigenvalues (p. 201). Also, the discussion of complex structure on p. 196 is somewhat nonstandard. We shall not be using any of these topics explicitly or regularly in the present course. However, we should note the material on that page starting with Theorem **3D** goes off in a much different direction and is worth understanding thoroughly, for it involves the simplest nontrivial cases of important concepts from linear algebra.

I.1 : Dot products

Suggested readings.

Ryan : pp. 8 – 11, 14, 86 – 88

Comments.

The first passage includes the beginning of Chapter **1** up to (but not including) the subheading "Lines," the second covers "Orthonormal pairs," and the third covers the subheading, "Orthonormal bases." Once again, some courses in linear algebra do not say much about dot (or inner) products, and if they contain very little about the subject it is unlikely they introduce the concept of an orthonormal set. However, such sets will play an important role in the present course, and they are discussed explicitly in the course notes.

I.2 : Cross products

Supplementary background readings.

Ryan : pp. 84 – 88, 136 – 138

Comments.

The first passage covers the beginning of Chapter $\mathbf{4}$ up to (but not including) the subheading, "Incidence geometry of the sphere," and the second covers the subheading "Cross products," which establishes further properties of this familiar operation on $\mathbf{3}$ – dimensional vectors. All of these topics are covered explicitly in the course notes. There will be a few instances when we shall use cross products, but we shall not use them as extensively as we shall use the dot products from the preceding section.

I. 3 : Linear varieties

Supplementary background readings.

Ryan : pp. 11 – 18

Comments.

This passage starts with the subheading, "The Euclidean plane," at the top of page 11 and continues through the subheading, "Parallel and intersecting lines," which ends at the bottom of page 18. Many of the ideas in this section of the notes are discussed for the special case of lines in the plane.

I.4 : Barycentric coordinates

Supplementary background readings.

Ryan : pp. 58, 68

Comments.

On page 58 there is a definition of barycentric coordinates, and one of the exercises on page 68 calls for a proof of one of the basic results on these numbers; the treatment of barycentric coordinates in the course notes is considerably more extensive.

Readings for Unit II from Ryan
(Vector algebra and Euclidean geometry)

II.1 : Approaches to Euclidean geometry

Supplementary background readings.

Ryan : pp. 5 – 15

Comments.

The passage begins with the subheading, "Three approaches to the study of geometry," on page 5, and it continues up to (but not including) the subheading, "Orthonormal pairs," on page 15. In the first few pages there is a discussion of different approaches to geometry that is somewhat complementary to the corresponding discussion in the notes, and the remaining material includes a discussion of the analytic approach in a manner similar to the course notes.

II.2 : Synthetic axioms of order and separation

Supplementary background readings.

Ryan : pp. 11 – 15, 50 – 52, 58 – 62, 68

Comments.

The first passage includes the algebraic formulation of the geometrical notion of betweenness for triples of collinear points, while the second passage is the subheading "Rays and angles," and its approach is somewhat more analytic than the approach in the course notes. In the third passage, barycentric coordinates are applied to study the Plane Separation Property and related results which play an important role in the course notes, and once again the treatment is somewhat more analytic in places. This passage ends just before the subheading, "Symmetries of a triangle." Finally, several exercises related to this section of the course notes are given on page 68.

II.3 : Measurement axioms

Supplementary background readings.

Ryan : pp. 11 – 15, 50 – 52, 58 – 62, 68

Comments.

The first passage contains basic definitions (as before), while the second passage, which is the subheading "Rays and angles," provides a different perspective on the theory of angle measurement. In the third passage, which starts with the subheading "Barycentric coordinates" and goes through the end of the subheading "Triangles," there is a discussion of sums of angles which complements the course notes. Page 68 contains exercises related to this section of the course notes.

II.4 : Congruence, superposition and isometries

Supplementary background readings.

Ryan : pp. 11 – 15, 19 – 21, 49 – 52, 55 – 62, 64 – 66, 68

Comments.

The comments for the previous section also apply here to the first passage, the subheadings "Barycentric coordinates," "Triangles" and "Congruence theorems for triangles" on pages 58 – 60, 61 – 62 and 65 – 66, and likewise to the exercises on the final page. In the other pages, the book studies the concept of ***isometry*** (which is introduced in the notes) from a somewhat different viewpoint, with particular emphasis on its ties to the intuitive notion of symmetry for certain familiar geometrical figures. A symmetry is an isometry that is a $1-1$ correspondence from a give figure to itself. In the subheading "Reflections," the appropriate symmetries are reflections of a plane about a line; physically speaking, these correspond to taking the mirror image of a point with respect to a specified line in the plane. The subheading "Symmetry groups" discusses an important symmetry property of isosceles triangles which is also covered in the notes but from a slightly different viewpoint. Another class of isometries — namely, the mathematical analog of a physical rotation — plays a very important role in Ryan's formulation of some key concepts involving angular measurement on pages 50 – 52 and 64 – 65. The subheadings, "Symmetries of a segment" and "Symmetries of an angle," which appear on pages 55 – 58, give a formal description of the intuitively apparent reflection symmetries of a closed line segment with respect to its midpoint and of an angle with respect to the line of it bisector. In the subheading "Affine symmetries" which appears on pages 49 – 50, there is a corresponding analysis involving affine transformations instead of isometries in certain cases. Finally, the addition and supplementary angle properties for angle measurements, which appear in the subheading "Addition of angles" on pages 60 – 61, are derived using rotations.

II.5 : Euclidean parallelism

Supplementary background readings.

Ryan : pp. 11 – 15, 17 – 18, 49 – 50, 58 – 59, 68

Comments.

The comments for Section **II.3** also apply here.

Readings for Unit III from Ryan
(Basic Euclidean concepts and theorems)

III.1 : Perpendicular lines and planes

Supplementary background readings.

Ryan : pp. 16 – 18

Comments.

This passage derives several key theorems in the notes and exercises using different approaches.

III.2 : Basic theorems on triangles

Supplementary background readings.

Ryan : pp. 62 – 64, 66 – 67

Comments.

The subheading "Angle sums for triangles" on pages 66 – 67 gives an alternate approach to the fundamental result in Euclidean geometry about the sums of the (measures of the) vertex angles of a triangle. On the other hand, the subheading "Symmetries of a triangle" on pages 62 – 64 takes a detailed look at a topic not covered in the notes; namely, describing the six affine self – symmetries of a triangle. If the triangle is an equilateral triangle, then these symmetries are in fact ***isometries*** (see Theorem 37), but otherwise some are not.

III.3 : Convex polygons

Supplementary background readings.

Ryan : pp. 75 – 81, 82 – 83

Comments.

These readings provide a slightly different approach to regular polygons which is considerably more algebraic than the notes, and they also contain some relevant exercises.

III.4 : Concurrence theorems

Supplementary background readings.

Ryan : pp. 54 – 55

Comments.

The treatment of the median theorem in this passage (from Ryan) corresponds to one of the exercises for the course notes.

III.5 : Similarity

Supplementary background readings.

Ryan : pp. 47 – 50, 55 – 58, 68

Comments.

The material on pages 47 – 49 gives a considerably more algebraic approach to similarities than the treatment in the notes, and the material on pages 55 – 58 describes the similarities of the plane which take a closed segment or angle to itself. There are some relevant exercises on page 68.

[There are no suggested readings in Ryan for Sections **III.6** or **III.7** of the notes.]

Readings for Unit V from Ryan
(Introduction to non – Euclidean geometry)

V.1 : Facts from spherical geometry

Supplementary background readings.

Ryan : pp. 84 – 123

Comments. The cited pages cover an entire chapter of Ryan's book (Chapter **4**), and the first few pages discuss the cross product of vectors in $\mathbf{R}^3$, a topic that is covered in Section **I.2** of the notes. This chapter discusses numerous aspects of spherical geometry which are not covered in the notes, and it also includes proofs of some results which are stated but not proved in the notes. Ryan's approach to spherical geometry is quite different from the treatment in the notes, and it is consistent with the treatment of Euclidean geometry in earlier sections of that book.

[There are no suggested readings in Ryan for Sections **V.2 — V.4** of the notes.]

V.5 : Further topics in hyperbolic geometry

Supplementary background readings.

Ryan : pp. 150 – 183

Comments.

Ryan's approach to hyperbolic geometry differs radically from the treatment in the notes; the former is strongly analytic while the latter is mainly synthetic. The treatment in Ryan provides a detailed and fairly comprehensive account of hyperbolic geometry, but even though the exposition does not use anything beyond the advanced undergraduate level, *the chapter is considerably more difficult to read than a typical advanced undergraduate mathematics text*. To indicate the difference in approach, we note that the line of inquiry that led to the discovery of hyperbolic geometry is not discussed at all, and also the key results of the subject (as given in the notes) are covered in the final third of the chapter in Ryan. From the viewpoint of the course notes, the first two thirds of the chapter may be viewed as the *construction of a formal mathematical model for the axioms defining a hyperbolic plane*; this model is often called the ***Lorentz model***, and it is named after H. A. Lorentz (1853 – 1928).

V.6 : Subsequent developments

Supplementary background readings.

Ryan : pp. 124 – 155

Comments.

The material in Ryan deals only with a few aspects of the topics covered in the notes. Pages 125 – 149 contain two entire chapters of the book, and they give a detailed account of the elliptic geometry discussed in the notes. Chapter **5** (pp. 124 – 140) discusses the incidence structure and its symmetries, while Chapter **6** (pp. 141 – 149) describes the measurement properties of elliptic geometry. This material is not as challenging to read as the material cited for the preceding section.

On pages 150 – 155 there is a construction of the Lorentz model for hyperbolic geometry which is connected to the standard relativistic model for space – time. This construction leads to some applications of hyperbolic geometry to relativistic space – time, but one cannot say that hyperbolic geometry is relativistic geometry or vice versa. We should note that the Lorentz model is often attributed to H. Poincaré, but it may have been known and used earlier in work of W. K. J. Killing (1847 – 1923) and K. T. W. von Weierstrass (1815 – 1897).

V.7 : Non – Euclidean geometry in modern mathematics

Supplementary background readings.

Ryan : pp. 150 – 183

Comments.

Again, the material in Ryan deals only with a few aspects of the topics covered in the notes. The construction of the Lorentz model on pages 150 – 155 is extremely relevant to this section. Also, the brief discussion of hyperbolic regular polygons on pages 176 – 177 of Ryan is closely related to the decompositions of the hyperbolic planes into regular polygons which are discussed extensively in the notes.