Examples

Linear systems of equations $\underset{m\times n}{A}\underset{n\times 1}{X}=\underset{m\times 1}{B}$

Write general solution in the form

$\vec{p}+\mathbb{V}$, $\vec{p}$ = particular solution

$\mathbb{V}$ = solutions to reduced eqn

$AX=0$. (Find a basis for $\mathbb{V}$)

1. $2x+3y=6$ defines line in $\mathbb{R}^2$

Matrix for system:

$$\begin{pmatrix} 2 & 3 & : & 6 \end{pmatrix}$$

Row Reduced Echelon form

$$\begin{pmatrix} 1 & \frac{3}{2} & : & 3 \end{pmatrix} \qquad x+\frac{3}{2}y=3$$

leading nonzero entry

y varies independently, $x=3-\frac{3}{2}y$.

Find $\vec{p}$ Set $y=0$ $\vec{p}=\begin{pmatrix}3\\0\end{pmatrix}$.

(NOT THE ONLY POSSIBLE CHOICE!!)

Find $\mathbb{V}$ Solve $AX=0$. Replace last column in prev. by 0, take echelon form $\begin{pmatrix} 1 & \frac{3}{2} & : & 0 \end{pmatrix}$ $\quad x+\frac{3}{2}y=0$

$$x=-\frac{3}{2}y.$$

Basic solution (AX=0) Set $y=1$ solve for x.

Basis for $\mathbb{V}$ is given by $\begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix}$, so

$\mathbb{V}=$ all vectors $t\cdot\begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix}$, where $t\in\mathbb{R}$.

General soln. $AX=B$ $\quad \overset{\vec{p}}{\begin{pmatrix} 3 \\ 0 \end{pmatrix}} + t\overset{\mathbb{V}}{\begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix}}$ $t\in\mathbb{R}$.

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2. $x+y+z=1$ defines plane in $\mathbb{R}^3$

$(A:B)$ is $\begin{pmatrix} 1 & 1 & 1 & : & 1 \end{pmatrix}$

leading entry variable

y & z vary independently, $x=1-y-z$

Find $\vec{p}$ Set $y=z=0$. $\vec{p}=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Find V Solve $AX=0$. $(1 \;\; 1 \;\; 1 : 0)$

$x=-y-z$. Basic solutions $\begin{cases} y=0, & z=1 \\ y=1, & z=0. \end{cases}$

The general solution to $AX=0$ is

$$\underbrace{u\begin{pmatrix}-1\\0\\1\end{pmatrix}+v\begin{pmatrix}-1\\1\\0\end{pmatrix}} \qquad u, v \in \mathbb{R}$$

$\underline{V}$ = all vectors ↑

General solution to $AX=B$

$$\underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}}_{\vec{p}} + \underbrace{u\begin{pmatrix}-1\\0\\1\end{pmatrix}+v\begin{pmatrix}-1\\1\\0\end{pmatrix}}_{\underline{V}} \qquad u, v \in \mathbb{R}$$

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3. $\left.\begin{aligned} x + y + z &= 4 \\ x - 2y + z &= 3 \end{aligned}\right\}$ define a line in $\mathbb{R}^3$

$(A:B) = \begin{pmatrix} 1 & 1 & 1 : 4 \\ 1 & -2 & 1 : 3 \end{pmatrix}$ Put into row reduced echelon form.

(Gauss-Jordan elimination)

$$\left(\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 1 & -2 & 1 & 3 \end{array}\right) \xrightarrow[\text{1st}]{\text{2nd minus}} \left(\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 0 & -3 & 0 & -1 \end{array}\right)$$

$$\xrightarrow[\text{2nd}]{-\frac{1}{3}\times} \left(\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & \frac{1}{3} \end{array}\right) \xrightarrow[\text{2nd}]{\text{1st minus}} \left(\begin{array}{ccc:c} 1 & 0 & 1 & 11/3 \\ 0 & 1 & 0 & 1/3 \end{array}\right)$$

leading entry columns

z varies independently, x & y depend on z.

$x = \frac{11}{3} - z \qquad y = \frac{1}{3}$ (indep. of z!)

Find $\vec{p}$ Set $z = 0$ $\vec{p} = \begin{pmatrix} \frac{11}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix}$.

Find $\underline{V}$ Solve system $(A:0)$ $\begin{cases} x = -z \\ y = 0 \end{cases}$

General solution to $AX=0$ is $t\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $t \in \mathbb{R}$.

General solution to $AX=B$ is

$$\underbrace{\begin{pmatrix} \frac{11}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix}}_{\vec{p}} + t \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{\underline{V}} \qquad t \in \mathbb{R}.$$

4. Same type as #3:

$$\left.\begin{aligned} x + 2y + z &= 6 \\ x - 2y + z &= 10 \end{aligned}\right\}$$ interse equation defines a line in $\mathbb{R}^3$

$$(A:B) = \begin{pmatrix} 1 & 2 & 3 & : & 6 \\ 1 & -2 & 1 & : & 10 \end{pmatrix}$$ Row reduce:

$$\begin{pmatrix} 1 & 2 & 3 & : & 6 \\ 1 & -2 & 1 & : & 10 \end{pmatrix} \xrightarrow[\text{1st}]{\text{2nd minus}} \begin{pmatrix} 1 & 2 & 3 & : & 6 \\ 0 & -4 & -2 & : & 4 \end{pmatrix} \xrightarrow[\text{2nd}]{-\frac{1}{4}\text{ times}}$$

$$\begin{pmatrix} 1 & 2 & 3 & : & 6 \\ 0 & 1 & \frac{1}{2} & : & -1 \end{pmatrix} \xrightarrow[2 \times \text{2nd}]{\text{1st minus}} \begin{pmatrix} 1 & 0 & 2 & : & 8 \\ 0 & 1 & \frac{1}{2} & : & -1 \end{pmatrix}$$

↑ ↑ leading columns.

z arbitrary, x & y depend on z.

Find $\vec{p}$ Set $z = 0$, get $\vec{p} = \begin{pmatrix} 8 \\ -1 \\ 0 \end{pmatrix}$

Find basis for V Reduced system $\begin{pmatrix} 1 & 0 & 2 & : & 0 \\ 0 & 1 & \frac{1}{2} & : & 0 \end{pmatrix}$, basis given by $z = 1$, so a basis vector is

$\begin{pmatrix} -2 \\ -\frac{1}{2} \\ 1 \end{pmatrix}$. General solution is

$$\begin{pmatrix} 8 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ -\frac{1}{2} \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$$