ADDITIONAL REMARKS ON CONCURRENCE

In an equilateral triangle, the centroid, circumcenter, orthocenter and incenter all coincide. The reason for this is simple: If we are given an equilateral triangle ΔABC and the midpoints of $[BC]$, $[AC]$ and $[AB]$ are given by D, E and F respectively, then

(i) the ray $[AD$ is also the angle bisector for $\angle BAC$, and the line AD is also the altitude from A to BC and the perpendicular bisector of $[BC]$ (all because $d(A,B) = d(A,C)$),
(ii) similarly for the rays $[BE$ and $[CF$ as well as the lines BE and CF.

However, in general the centroid, circumcenter, orthocenter and incenter of a triangle are all distinct, and in fact one can prove that if two of these points coincide then the triangle is equilateral. The proof splits into six cases corresponding to the following hypotheses:

(1) The centroid and orthocenter coincide.
(2) The centroid and incenter coincide.
(3) The centroid and circumcenter coincide.
(4) The incenter and orthocenter coincide.
(5) The circumcenter and incenter coincide.
(6) The circumcenter and orthocenter coincide.

Preliminaries

Before proving the theorem stated above, we shall establish some auxiliary results that we shall need.

PROPOSITION. *If L is a line, x is a positive real number, and X is a point not on L, then there are at most two points on L whose distance from X is equal to x.*

Proof. Suppose that $B,\ C,\ D$ are three points on L such that $d(X,B) = d(X,C) = d(X,D) = x$. Relabeling the points if necessary, we may assume that $B * C * D$ holds. Let E be the midpoint of $[BC]$ and let F be the midpoint of $[CD]$. Since X is equidistant from $B,\ C,\ D$ it follows that XE is the perpendicular bisector of $[BC]$ and XF is the perpendicular bisector of $[CD]$. However, we know that there is only one perpendicular from X to L, so this is a contradiction. The source of this contradiction is our assumption that there are three points on L which are equidistant from X, and therefore we conclude that there are at most two such points.∎

The second result analyzes pairs of triangles which satisfy **SSA**; as noted in Section II.4, there is no general congruence theorem in this case, but the following result shows that there are at most two possibilities:

THEOREM. (**SSA** congruence ambiguity). (i) *Suppose that we are given ΔABC. Then there is at most one point $G \in (AC$ such that $G \neq C$ and $d(B,C) = d(B,G)$.*

(ii) *If G is given as above and ΔDEF is a triangle such that $d(A,B) = d(E,F)$, $d(B,C) = d(E,F)$ and $|\angle BAC| = |\angle EDF|$, then either $\Delta ABC \cong \Delta DEF$ or else $\Delta ABG \cong \Delta DEF$.*

Proof. (i) There is at most one other point $G \in AC$ such that $d(B,G) = d(B,C)$ by the preceding proposition, so there is at most one such point on $(AC$.∎

(ii) Let $H \in (AC$ be such that $d(A,H) = d(D,F)$. Then by **SAS** we have $\Delta ABH \cong \Delta DEF$, and consequently we also have that $d(B,H) = d(E,F) = d(B,C)$. If $H = C$, then $\Delta ABC \cong \Delta DEF$; on the other hand, if $H \neq C$, then H must be the second point G which satisfies the conditions in (i), and we have $\Delta ABG \cong \Delta DEF$.■

Remark. If $BC \perp AC$, then the only point $G \in AC$ such that $d(B,G) = d(B,C)$ is C itself, and this is why one has a hypotenuse-side congruence theorem for right triangles. On the other hand, it is also possible that one has a second point $G \in AC$ at the prescribed distance but G does not lie on the open ray $(AC$; this happens if $d(B,C) > d(A,B)$. On the other hand, if we are given ΔABC such that $d(A,B) > d(A,C)$ and AC is not perpendicular to BC, then there is a point $G \neq C$ on $(BC$ such that $d(A,G)$ and $d(A,C)$, which means that ΔABC and ΔABG are not congruent even though they satisfy **SSA**.

Proofs in the six individual cases

In all cases it will suffice to prove that $d(A,B) = d(A,C)$, for if we know this we can also conclude that $d(A,C) = d(C,A) = d(C,B) = d(B,C)$ by switching the roles of C and A in the appropriate discussion.

One fact which is used repeatedly in our arguments is that the incenter and centroid of a triangle can never lie on the triangle itself; in contrast, it is possible for the circumcenter or orthocenter to lie on the triangle, and in fact they always do so for right triangles.

CASE (1): *The centroid and orthocenter of ΔABC coincide.* Let D be the midpoint of $[AC]$. If the centroid and orthocenter are the same, then AD also contains the orthocenter; but this means that $AD \perp BC$. Therefore we have $d(B,D) = d(C,D)$, $|\angle ADB| = 90^\circ = |\angle ADC|$, and $d(A,D) = d(A,D)$, so that $\Delta ADB \cong \Delta ADC$ by **SAS**. By the conclusion of the preceding sentence it follows that $d(A,B) = d(A,C)$.■

CASE (2): *The centroid and incenter of ΔABC coincide.* We shall give two proofs; the first does not require the use of Playfair's Postulate, but the second does.

First proof. We shall suppose that $d(A,B) \neq d(A,C)$ and derive a contradiction. Without loss of generality we may assume that $d(A,B) < d(A,C)$ (we can dispose of the other case by switching the roles of B and C in the argument that follows). Let D be the midpoint of $[BC]$; since the centroid and incenter coincide, we know that $[BD$ is the angle bisector of $\angle BAC$.

Notice that ΔABD and ΔADC satisfy the **SSA** conditions $|\angle BAD| = |angleBAC|$, $d(A,D) = d(A,D)$ and $d(B,D) = d(C,D)$. By itself this is not enough to prove that $\Delta ADB \cong \Delta ADC$, which in our setting would be equivalent to showing that $d(A,B) = d(A,C)$, so we really need to determine whether $d(A,B) < d(A,C)$ is possible. — Let $E \in (AB$ be such that $d(A,E) = d(A,C)$; the distance inequality implies that $A * B * E$ must hold. We then have $\Delta DAC \cong \Delta DAE$ by **SAS**. This means that $d(D,E) = d(D,C) = d(D,B)$, where the second inequality holds because D is the midpoint of $[BC]$. By the Isosceles Triangle Theorem it follows that $|\angle DEB = \angle DEA|$ is equal to $|\angle DBE = \angle CBE|$, and by the previously established congruence relation we know that $|\angle DEA| = |\angle ACB|$. If we combine these, we see that $|\angle EBC| = |\angle ACB|$

On the other hand, the Exterior Angle Theorem implies that $|\angle EBC| > |\angle ACB|$, so we have a contradiction. The source of this contradiction is our assumption that $d(A,B)$ and $d(A,C)$ are unequal, and therefore we must have $d(A,B) = d(A,C)$.■

Second proof. This uses the Angle Bisector Theorem from Section III.5 of the notes (which depends upon the theory of similar triangles and hence upon Playfair's Postulate). — Let D be the midpoint of $[AC]$. Since the centroid and incenter are the same and neither lies on $[AC]$, it follows that $[AD$ is the angle bisector for $\angle BAC$. Therefore the Angle Bisector Theorem implies that

$$\frac{d(A,C)}{d(C,B)} \quad = \quad \frac{d(A,D)}{d(D,C)}$$

and since D is the midpoint of $[AC]$ the right hand side is equal to 1, which in turn implies that $d(A,C)$ must be equal to $d(B,C)$.■

CASE (3): *The centroid and circumcenter of ΔABC coincide.* Let D be the midpoint of $[AC]$. If the centroid and circumcenter are the same, then AD also contains the circumcenter, and since the centroid does not lie on the triangle it follows that the same is true for the circumcenter; this means that AD is the perpendicular bisector of $[BC]$. Therefore we must have $d(A,B) = d(A,C)$.■

CASE (4): *The incenter and orthocenter of ΔABC coincide.* Let $[AX$ be the bisector of $\angle BAC$, and let $D \in (AX \cap (BC)$ be the point whose existence is guaranteed by the Crossbar Theorem. If the incenter and orthocenter are the same, then AD also contains the orthocenter, and as in the first case this means that $AD \perp BC$. Therefore we have $|\angle ADB| = 90° = |\angle ADC|$, $d(A,D) = d(A,D)$, and $|\angle BAD| = |\angle CAD|$, so that $\Delta ADB \cong \Delta ADC$ by **ASA**. By the conclusion of the preceding sentence it follows that $d(A,B) = d(A,C)$.■

CASE (5): *The circumcenter and incenter of ΔABC coincide.* Let D be the midpoint of $[BC]$, and let E be the point given by the circumcenter and incenter. Let F and G denote the feet of the perpendiculars from E to AB and AC respectively. Since $[AE$ bisects $\angle BAC$ we know that $F \in (AB$ and $G \in (AC$. Also, since $[BE$ bisects $\angle ABC$ and $[CE$ bisects $\angle ACB$, we also know that $F \in (BA$ and $G \in (CA$, so that $F \in (AB)$ and $G \in (AC$.

By the characterization of angle bisectors in Section III.4, we know that $d(E,A) = d(E,B) = d(E,C)$, so by **AAS** we also have $\Delta EFA \cong \Delta EGA$, so that $d(A,F) = d(A,G)$. On the other hand, since E is also the circumcenter of ΔABC we also have $d(E,A) = d(E,B)$, and the Hypotenuse-Side Congruence Theorem for right triangles then shows that $\Delta EFB \cong \Delta EGC$. The latter implies that $d(F,B) = d(G,C)$. Combining this with the previous observations we find that

$$d(A,B) \quad = \quad d(A,F) \ + \ d(F,B) \quad = \quad d(A,G) \ + \ d(G,C) \quad = \quad d(A,C)$$

which is what we needed to prove.■

CASE (6): *The circumcenter and orthocenter of ΔABC coincide.*

Let D be the midpoint of $[BC]$, and let M be the unique line through A which is perpendicular to BC. If the circumcenter and orthocenter are the same, then the circumcenter G must lie on M. But G also lies on the perpendicular bisector of $[BC]$, so either M is this perpendicular bisector or else $G = D$.

Suppose that D is not the circumcenter G. Since the line joining D to the circumcenter is perpendicular to BC and the line joining A to the orthocenter is perpendicular to BC, it follows that these lines must coincide. But if this happens then A lies on the perpendicular bisector of $[BC]$, and therefore we must have $d(A,B) = d(A,C)$.

To complete the proof we need to show that D cannot be the circumcenter of the triangle if the circumcenter and orthocenter coincide. If this happens, then D is also the orthocenter, which means that the line $BC = BD$ is perpendicular to both AB and AC. This is impossible because there is only one perpendicular to BC which contains the external point A.$\blacksquare$

The role of Playfair's Postulate

The following discussion assumes material from Unit V of the notes.

In Section III.4 of the notes, Euclidean geometry was the setting for the proofs of the four triangle concurrence theorems. Two of these results are also true in hyperbolic geometry. The proof of the incenter theorem carries over with no essential changes, and there is also a centroid theorem, but its proof is entirely different from the Euclidean argument. A sketch of the proof for hyperbolic geometry appears in Exercise K–19 on page 226 of the following book:

> M. J. Greenberg, *Euclidean and Non-Euclidean Geometries — Development and History* (Second Edition). W. H. Freeman, San Francisco, 1980. [*Note:* The most recent Fourth Edition appeared in 2007.]

One reason for including the first proof of Case (2) is that it does not require Playfair's Postulate and hence is also valid in hyperbolic geometry. In all the remaining cases, the proofs are valid in neutral geometry **provided** that the given points exist.

The preceding sentence suggests that the circumcenter theorem and orthocenter theorem do not extend to hyperbolic geometry; for each statement, one can construct triangles in the hyperplane for which the given lines are not concurrent. However, even though the circumcenter theorem fails to hold in hyperbolic geometry, one does have the following result in that setting:

THEOREM. *Suppose we are given ΔABC in a hyperbolic plane $\mathbf{P}$, and let L, M and N be the perpendicular bisectors of the sides. Then exactly one of the following is true:*

(a) *The lines L, M and N are concurrent (as is always the case in Euclidean geometry).*

(b) *The lines L, M and N have a common perpendicular.*

(c) *The lines L, M and N are triply asymptotic; in other words, the lines are pairwise disjoint, but there are ruler functions $f : L \to \mathbf{R}$, $g : M \to \mathbf{R}$ and $h : N \to \mathbf{R}$ for these lines such that*

$$\lim_{t\to\infty} d\big(\, f^{-1}(t),\, g^{-1}(t)\,\big) \;\;=\;\; \lim_{t\to\infty} d\big(\, g^{-1}(t),\, h^{-1}(t)\,\big) \;\;=\;\; 0\ .$$

The concept of asymptotic parallels is discussed at the end of Section V.4 in the notes. Observe that the two conditions in (c) also imply that

$$\lim_{t\to\infty} d\big(\, f^{-1}(t),\, h^{-1}(t)\big) \;\;=\;\; 0\ .$$

THE EULER LINE AND HYPERBOLIC GEOMETRY. In contrast to Euclidean geometry, even if a hyperbolic triangle has both a circumcenter and orthocenter, it does not follow that these points and the centroid are collinear. An example (using the Poincaré disk model for the hyperbolic plane) is described in the following article:

`http://josm.geneseo.edu/1-1/00-03.pdf`