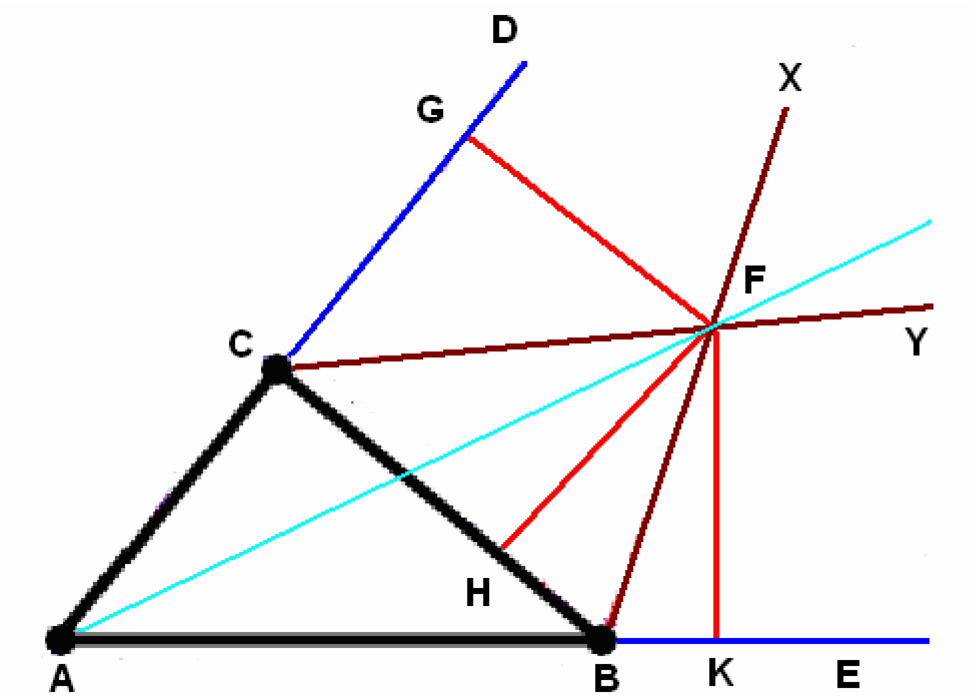
The excenters of a triangle

This material supplements Section III.4 *of the notes.*

Theorem. *Given a triangle* △**ABC,** *let* **D** *and* **E** *be points such that* **A∗C∗D** *and* **A∗B∗E,** *let* **[BX** *and* **[CY** *denote the bisectors of* ∠**ABE** *and* ∠**BCD** *respectively (the* ***external bisectors*** *of the angles at* **B** *and* **C**), *and let* **[AZ** *denotes the bisector of* ∠**BAC.** *Then the rays* **[BX, [CY** *and* **[AZ** *are concurrent.*

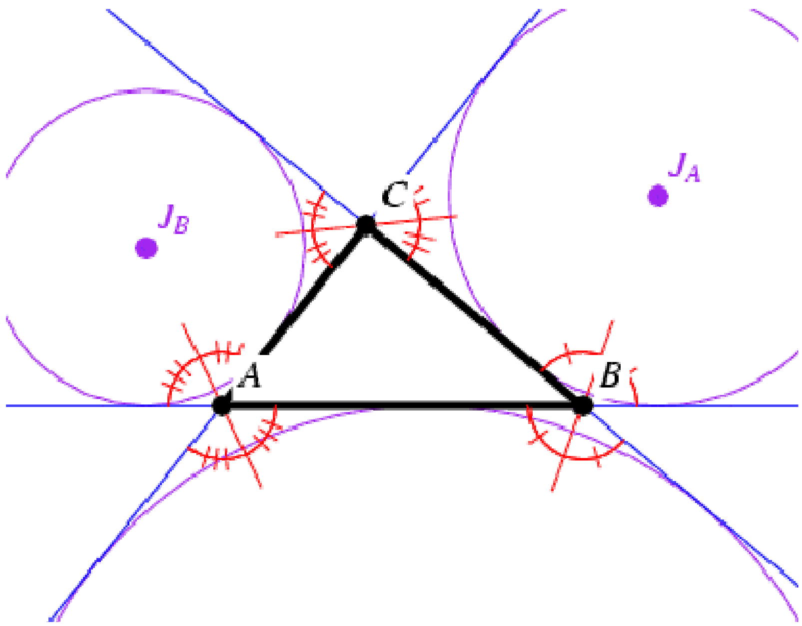


Proof. We shall first verify that **[BX** and **[CY** meet at a point **F,** and this point lies on the side of **BC** opposite **A.** The bisector conditions imply that |∠**XBC**| = ½ |∠**EBC**| and |∠**YCB**| = ½ |∠**GCB**|. Since both |∠**EBC**| and |∠**GCB**| are less than 180 degrees, it follows that |∠**XBC**| + |∠**YCB**| = ½ |∠**EBC**| + ½ |∠**GCB**| < 180, and therefore by the classical version of Euclid's Fifth Postulate it follows that **(BX** and **(CY** meet at a point which lies on the same side of **BC** as **Y** and **Z.** To see that this intersection point **F** lies on the side of **BC** opposite **A,** proceed as follows: The bisector condition implies that **F, X** and **E** lie on the same side of **BC,** while the betweenness condition implies that **A** and **E** lie on opposite sides of **BC,** and therefore **F** and **A** must lie on opposite sides of **BC.**

We claim that **F** lies in the interior of ∠**BAC.** Since **F** lies on **(BX** and **(CY,** it follows that **F** lies in the interiors of both ∠**ABE** and ∠**BCD.** The second of these implies that **F** and **B** lie on the same side of **CD** = **AC,** while the first implies that **F** and **C** lie on the same side of **BE** = **AB;** combining these, we obtain the assertion in the first sentence of this paragraph.

By construction the point **F** does not lie on any of the lines **AB, BC** or **AC.** Let **G, H** and **K** denote the feet of the perpendiculars from **F** to **AC, BC** and **AB** respectively. By the characterization of angle bisectors, we know that d(**F, G**) = d(**F, H**) and d(**F, H**) = d(**F, K**). Since **F** lies in the interior of $\angle$ **BAC,** the preceding equations imply that **F** lies on the angle bisector **[AZ** of that angle. Therefore the point **F,** which by construction lies on **(BX** and **(CY,** must also lie on **(AZ,** which is what we wanted to prove.■

The point **F** is called an <u>**excenter**</u> for the triangle. If we interchange the roles of the vertices in the original triangle, we obtain three different excenters. In the drawing below, the points $\mathbf{J_A}$ and $\mathbf{J_B}$ are two of the excenters, and the red lines through **A** and **B** meet in the third excenter $\mathbf{J_C}$, which is out of the picture:



(***Source***: http://mathworld.wolfram.com/ExteriorAngleBisector.html)