I.5 : Some examples

In this section we shall work a few problems which illustrate the concepts discussed in Sections **I.3 – I.4** and are similar to homework exercises for the latter sections.

PROBLEM 1. *Let* **L** *be the line in* $\mathbf{R}^3$ *joining the points* **a** = **(1, 0, 0)** *and* **b** = **(0, 1, 0),** *and let* **M** *be the line joining points* **c** = **(1, 2, 0)** *and* **d** = **(0, 0, 1).** *Determine whether the lines intersect, and if so find their point of intersection***.**

SOLUTION. The lines intersect if and only if we can find scalars u and v such that

$$\mathbf{a} + u(\mathbf{b}-\mathbf{a}) = \mathbf{c} + v(\mathbf{d}-\mathbf{c})$$

or equivalently if we can write

$$\mathbf{c}-\mathbf{a} = u(\mathbf{b}-\mathbf{a}) - v(\mathbf{d}-\mathbf{c})$$

for suitable scalars u and v. Now we have

$$\mathbf{b}-\mathbf{a} = \begin{pmatrix}-1\\1\\0\end{pmatrix}, \qquad \mathbf{d}-\mathbf{c} = \begin{pmatrix}-1\\-2\\1\end{pmatrix}, \qquad \mathbf{c}-\mathbf{a} = \begin{pmatrix}0\\2\\0\end{pmatrix}$$

so the answers reduce to finding solutions for the system of three equations $u - v = 0$, $u + 2v = 2$, and $v = 0$. This system is ***inconsistent***, and therefore ***the two lines do not have any points in common.*** ■

PROBLEM 2. *Let* **L** *be the line in* $\mathbf{R}^3$ *joining the points* **a** = **(0, 0, 1)** *and* **b** = **(1, 1, 3),** *and let* **M** *be the line joining points* **c** = **(2, 1, 4)** *and* **d** = **(1, 2, 4).** *Determine whether the lines intersect, and if so find their point of intersection***.**

SOLUTION. The intersection and solution criteria are the same as before. We now have the vector identities

$$\mathbf{b}-\mathbf{a} = \begin{pmatrix}1\\1\\2\end{pmatrix}, \qquad \mathbf{d}-\mathbf{c} = \begin{pmatrix}-1\\1\\0\end{pmatrix}, \qquad \mathbf{c}-\mathbf{a} = \begin{pmatrix}2\\1\\3\end{pmatrix}$$

so in this case the answer reduces to finding solutions for the system of linear equations $u + v = 2$, $u - v = 1$, and $2u = 3$. In this case one has a unique solution; namely, $u = 3/2$ and $v = ½$. Direct substitution of these values for u and v then yields the desired common point, which is **(3/2, 3/2, 4).** ■

PROBLEM 3. *Describe the line determined by the intersections of the planes with equations* $x + y + z = 1$ *and* $x + 2y + 3z = 6$.

SOLUTION. We shall analyze the set of solutions using linear algebra and the augmented matrix of coefficients:

$$\begin{pmatrix} 1 & 1 & 1 & : & 1 \\ 1 & 2 & 3 & : & 6 \end{pmatrix}$$

If we put this into row reduced echelon form, we obtain a new system of equations which is equivalent to the old one in which two variables are given explicitly as first degree polynomials in the third. Subtracting the first row from the second yields the matrix

$$\begin{pmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 2 & : & 5 \end{pmatrix}$$

and if we now subtract the second row from the first we obtain

$$\begin{pmatrix} 1 & 0 & -1 & : & -4 \\ 0 & 1 & 2 & : & 5 \end{pmatrix}$$

which leads to the equations $x = z - 4$ and $y = 5 - 2z$. If we add the tautological equation $z = z$ to this list we obtain the parametric equations for the line, which can be rewritten in vector form as

$$\mathbf{s}(z) = (-4, 5, 0) + z\,(1, -2, 1).$$

Taking $z = 1$, we see that this is the line joining $(-4, 5, 0)$ to $(-3, 3, 1)$. — As always, it is reassuring to check the correctness of the calculations by verifying that each of these two points does satisfy the equations for both planes.■

PROBLEM 4 (Special case of Ceva's Theorem). *Suppose we are given noncollinear points* **A, B, C** *in* $\mathbf{R}^2$, *suppose they are the vertices of an isosceles triangle such that* $d(\mathbf{A}, \mathbf{B}) = d(\mathbf{A}, \mathbf{C})$, *and let* **D** *be the midpoint of* **[BC]**. *Suppose also that we have a pair of points* **E** *and* **F** *on the segments* **[AC]** *and* **[AB]** *respectively such that* $d(\mathbf{C}, \mathbf{E}) = (1/3)\, d(\mathbf{C}, \mathbf{A})$ *and that* $d(\mathbf{B}, \mathbf{F}) = (1/3)\, d(\mathbf{B}, \mathbf{A})$. *Assume further that the lines* **BE** *and* **CF** *meet at some point* **G** *on the line segment* **[AD]**. *Find the ratio* $d(\mathbf{D}, \mathbf{G}) / d(\mathbf{D}, \mathbf{A})$.

SOLUTION. In these problems it is always good to start by writing out everything that is given. First, we know that the midpoint **D** satisfies $\mathbf{D} = \frac{1}{2}\mathbf{B} + \frac{1}{2}\mathbf{C}$. Next, we know that $\mathbf{E} = p\mathbf{A} + (1-p)\mathbf{C}$ and $\mathbf{F} = q\mathbf{A} + (1-q)\mathbf{B}$; we can conclude that $p = q = 1/3$ from the assumption on distance ratios and the vector equations $\mathbf{E} - \mathbf{C} = p(\mathbf{A} - \mathbf{C})$ and $\mathbf{F} - \mathbf{C} = q(\mathbf{B} - \mathbf{C})$. Also, since the point **G** lies on **BE** and **CF** it follows that we have $\mathbf{G} = u\mathbf{B} + (1-u)\mathbf{E} = v\mathbf{C} + (1-v)\mathbf{F}$. We may now use the formulas for **E** and **F** to obtain the following expressions for **G**:

$$\mathbf{G} = u\mathbf{B} + [(1-u)/3]\mathbf{A} + [2(1-u)/3]\mathbf{C} = v\mathbf{C} + [(1-v)/3]\mathbf{A} + [2(1-v)/3]\mathbf{B}$$

In both of these expressions the coefficients of **A, B,** and **C** add up to 1, and since **A, B,** and **C** are noncollinear the results on barycentric coordinates imply the corresponding coefficients are equal. Thus we have the following equations:

$$(1 - u)/3 \;=\; (1 - v)/3$$
$$u \;=\; 2(1 - v)/3$$
$$v \;=\; 2(1 - u)/3$$

The first of these equations implies $u = v$ and the others then yield $u = 2/5$. If we substitute this into the formula for **G** we see find that

$$\mathbf{G} \;=\; (1/5)\,\mathbf{A} + (2/5)\mathbf{B} + (2/5)\mathbf{C} \;=\; (1/5)\,\mathbf{A} + (4/5)\mathbf{D}$$

so that $\mathbf{G} - \mathbf{D} = (1/5)\,(\mathbf{A} - \mathbf{D})$ and hence the ratio $d(\mathbf{D}, \mathbf{G}) / d(\mathbf{D}, \mathbf{A})$ is equal to $1/5$.■