ADDITIONAL PROPERTIES OF BETWEENNESS

NOTE. *The following should be inserted before the subheading* Separation axioms *on page 47 of the file* `geometrynotes2a.pdf`*.*

Here are some additional facts about betweenness with vector proofs (however, many computational steps have been omitted).

Example 1. *Suppose that* $\mathbf{a} \neq \mathbf{b}$ *and for* $i = 1, 2, 3$ *we have* $\mathbf{x}_i = a + t_i(\mathbf{b} - \mathbf{a})$ *where* $t_i \in \mathbf{R}$*. If* $t_1 < t_2 < t_3$*, then we have the ordering relation* $\mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3$*.*

SOLUTION. The distance between $\mathbf{x}_i$ and $\mathbf{x}_j$ is equal to $|\mathbf{x}_i - \mathbf{x}_j|$, and if we substitute the expressions for the vectors in question we see that

$$|\mathbf{x}_i - \mathbf{x}_j| \quad = \quad |t_i - t_j| \cdot |\mathbf{b} - \mathbf{a}| \ .$$

Therefore we have

$$|\mathbf{x}_1 - \mathbf{x}_3| \quad = \quad (t_3 - t_1) \cdot |\mathbf{b} - \mathbf{a}| \quad = \quad (t_3 - t_2) \cdot |\mathbf{b} - \mathbf{a}| \ + \ (t_2 - t_1) \cdot |\mathbf{b} - \mathbf{a}| \quad =$$

$$|\mathbf{x}_2 - \mathbf{x}_3| \ + \ |\mathbf{x}_1 - \mathbf{x}_2|$$

and therefore by the definition of betweenness we have $\mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3$.■

Example 2. *Suppose that* $\mathbf{a}*\mathbf{b}*\mathbf{c}$*. Then we have the inclusion of closed rays* $[\mathbf{bc} \ \subset \ [\mathbf{ac}$*.*

SOLUTION. By hypothesis we have $\mathbf{b} \ = \ \mathbf{a}+t(\mathbf{c}-\mathbf{a})$ for some scalar t satisfying $0 < t < 1$. If $\mathbf{x} \in [\mathbf{bc}$, then $\mathbf{x} \ = \ \mathbf{b} + u(\mathbf{c} - \mathbf{b})$ for some scalar u satisfying $u \geq 0$. Substituting the expression for $\mathbf{b}$ in the preceding equation, we find that

$$\mathbf{x} \quad = \quad \mathbf{a} + (t + u - tu) \cdot (\mathbf{c} - \mathbf{a}) \ .$$

To show that $\mathbf{x} \in [\mathbf{ac}$, we need to check that the coefficient of $\mathbf{c} - \mathbf{a}$ is positive. However, we have $t + u - tu = u(1 - t) + t$; the first term on the right hand side is a product of nonnegative factors and hence is nonnegative, and since $t > 0$ it follows that the entire expression is positive.■

Example 3. *Suppose that* $\mathbf{a} * \mathbf{b} * \mathbf{c}$ *and* $\mathbf{b} * \mathbf{x} * \mathbf{c}$*. Then we have* $\mathbf{a} * \mathbf{x} * \mathbf{c}$*.*

SOLUTION. This is similar to the preceding example. By assumption we have

$$\mathbf{b} \ = \ \mathbf{a} + t(\mathbf{c} - \mathbf{a}) \qquad \mathbf{x} \ = \ \mathbf{b} + u(\mathbf{c} - \mathbf{b})$$

where $0 < t, u < 1$. It follows that

$$\mathbf{x} \quad = \quad \mathbf{a} + (t + u - tu) \cdot (\mathbf{c} - \mathbf{a}) \ .$$

The desired conclusion will follow if we can verify that the coefficient of $\mathbf{c} - \mathbf{a}$ lies strictly between 0 and 1. Now

$$(1-t)\,(1-u) \quad = \quad 1 - u - t + tu$$

and since the left hand side lies between 0 and 1 it follows that the same is true for the right hand side. But this means that

$$u + t - tu \quad = \quad 1 \;-\; (1 - u - t + tu)$$

must also lie strictly between 0 and 1.■

Here is a final example along the lines of the first one.

Example 4. *Suppose that* $\mathbf{x}$ *and* $\mathbf{y}$ *are distinct points such that* $\mathbf{a} * \mathbf{x} * \mathbf{c}$ *and* $\mathbf{a} * \mathbf{y} * \mathbf{c}$ *are both true. Then either* $\mathbf{a} * \mathbf{x} * \mathbf{y}$ *or* $\mathbf{a} * \mathbf{y} * \mathbf{x}$ *is true.*

SOLUTION. We may write

$$\mathbf{x} \quad = \quad \mathbf{a} + t(\mathbf{c} - \mathbf{a}) \qquad \mathbf{y} \quad = \quad \mathbf{a} + u(\mathbf{c} - \mathbf{a})$$

where $0 < u, t < 1$. Clearly we also have

$$\mathbf{a} \quad = \quad \mathbf{a} + 0(\mathbf{c} - \mathbf{a})\ .$$

Either $0 < t < u$ or $0 < u < t$ is true. In the first case we have $\mathbf{a} * \mathbf{x} * \mathbf{y}$, and in the second case we have $\mathbf{a} * \mathbf{y} * \mathbf{x}$.■