VECTOR GEOMETRY AND BASIC PROPERTIES OF PARAELLOGRAMS

NOTE. *The following should be inserted after the proof of Proposition III.3.6 on page 107 of the file* `geometrynotes3a.pdf`*.*

In the notes there are synthetic proofs for several basic theorems about parallelograms. Our purpose here is to give some alternate proofs using vectors.

REVIEW. By Exercise I.4.3 in the file `math133exercises1.pdf`, if we are given noncollinear points $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{d}$ in $\mathbf{R}^2$ and $\mathbf{c} = \mathbf{b} + \mathbf{d} - \mathbf{a}$, then the four points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ (in that order) form the vertices of a parallelogram.■

This observation has two simple but important consequences:

$$\mathbf{b} - \mathbf{a} = \mathbf{c} - \mathbf{d}\,, \qquad \mathbf{c} - \mathbf{b} = \mathbf{d} - \mathbf{a}$$

These follow directly from the formula for $\mathbf{c}$ given above, and they immediately imply all the basic measurement properties of a parallelogram:

If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *and* $\mathbf{d}$ *(in that order) form the vertices of a parallelogram, then the following hold:*

(i) $d(\mathbf{a}, \mathbf{b}) = d(\mathbf{c}, \mathbf{d})$
(ii) $d(\mathbf{a}, \mathbf{d}) = d(\mathbf{b}, \mathbf{c})$
(iii) $|\angle\, \mathbf{dab}| = |\angle\, \mathbf{bcd}|$
(iv) $|\angle\, \mathbf{dab}| + |\angle\, \mathbf{adc}| = 180^\circ$

Derivations. Since $\mathbf{b} - \mathbf{a} = \mathbf{c} - \mathbf{d}$ and $\mathbf{c} - \mathbf{b} = \mathbf{d} - \mathbf{a}$, we have the following:

$$d(\mathbf{a}, \mathbf{b}) = |\mathbf{b} - \mathbf{a}| = |\mathbf{c} - \mathbf{d}| = d(\mathbf{c}, \mathbf{d})$$

$$d(\mathbf{a}, \mathbf{d}) = |\mathbf{d} - \mathbf{a}| = |\mathbf{b} - \mathbf{c}| = d(\mathbf{b}, \mathbf{c})$$

$$\cos\, |\angle\, \mathbf{dab}| = \frac{(\mathbf{d} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a})}{|\mathbf{d} - \mathbf{a}| \cdot |\mathbf{b} - \mathbf{a}|} = \frac{(\mathbf{b} - \mathbf{c}) \cdot (\mathbf{d} - \mathbf{c})}{|\mathbf{b} - \mathbf{c}| \cdot |\mathbf{d} - \mathbf{c}|} = \cos\, |\angle\, \mathbf{bcd}|$$

Since the cosine function is a strictly decreasing function on angle measurements between 0 and 180 degrees, the third line implies (iii). Of course, the first two lines imply (i) and (ii). Finally, to prove (iv) we proceed much as in (iii), but the outcome is slightly different:

$$\cos\, |\angle\, \mathbf{adc}| = \frac{(\mathbf{a} - \mathbf{d}) \cdot (\mathbf{c} - \mathbf{d})}{|\mathbf{a} - \mathbf{d}| \cdot |\mathbf{c} - \mathbf{b}|} = -\frac{(\mathbf{a} - \mathbf{d}) \cdot (\mathbf{c} - \mathbf{d})}{-|\mathbf{a} - \mathbf{d}| \cdot |\mathbf{c} - \mathbf{b}|} =$$

$$-\frac{(\mathbf{d} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a})}{|\mathbf{d} - \mathbf{a}| \cdot |\mathbf{b} - \mathbf{a}|} = -\cos\, |\angle\, \mathbf{dab}|$$

Since $\cos\theta \;\; = \;\; -\cos\varphi$ if and only if θ and φ are supplementary, the conclusion of (iv) follows immediately.$\blacksquare$

Here is one more standard result on parallelograms:

If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *and* $\mathbf{d}$ *(in that order) form the vertices of a parallelogram, then the diagonal lines* $\mathbf{ac}$ *and* $\mathbf{bd}$ *intersect at a point which is the midpoint of both* $[\mathbf{ac}]$ *and* $[\mathbf{bd}]$ *(in words, the diagonals of the parallelogram bisect each other).*

To prove this it is only necessary to check that $\frac{1}{2}(\mathbf{a}+\mathbf{c})$ and $\frac{1}{2}(\mathbf{b}+\mathbf{d})$ are the same point. This can be done by noting that the formula for $\mathbf{c}$ implies

$$\frac{1}{2}(\mathbf{a}+\mathbf{c}) \;\; = \;\; \frac{1}{2}\Big(\,\mathbf{a}+(\mathbf{b}+\mathbf{d}-\mathbf{a})\Big) \;\; = \;\; \frac{1}{2}(\mathbf{b}+\mathbf{d}) \;.\blacksquare$$