$\mathbf{III.7}$: Areas and volumes

Most upper level undergraduate textbooks in geometry do not cover these topics, but for the sake of completeness we shall explain how they fit into the setting of these notes. Our axioms for area theory will be adapted from those presented by the School Mathematics Study Group in the following reference:

> School Mathematics Study Group, ***Mathematics for High School: Geometry, Parts 1 and 2 (Student Text)***, Yale University Press, New Haven and London, 1961.

Here are some online references that may also be helpful:

http://en.wikipedia.org/wiki/Area_(geometry)

http://www.gomath.com/htdocs/ToGoSheet/Geometry/area.html

http://en.wikipedia.org/wiki/Area

When we think of areas in the plane, we generally think of them as being defined for certain types of sets called ***closed regions***; in particular, they generally have boundaries given by reasonable curves and contain these boundary curves. For example, a closed triangular region should consist of some triangle $\triangle$**ABC** along with its interior, and similarly for other convex polygons. It will be convenient to make this intuitive notion precise.

Definition. Suppose that $\mathbf{A}_1, \dots, \mathbf{A}_n$ are the vertices of a convex polygon (taken in that order). The ***closed region bounded by the convex polygon*** (or its ***closed convex polygonal region***) is the intersection of the closed half planes $\mathbf{H}^{\#}(\mathbf{A}_k\mathbf{A}_{k+1}; \mathbf{A}_{k+2})$, with the numbering conventions of Section $\mathbf{III.3}$. We shall often say that the convex polygon is the ***boundary*** of the closed convex polygonal regions and that the closed region is ***bounded*** by the polygon. We shall denote the closed convex polygonal region associated to $\mathbf{A}_1 \dots \mathbf{A}_n$ (in that order) by $\blacklozenge\mathbf{A}_1 \dots \mathbf{A}_n$.

The following result will be helpful for deriving area formulas:

Theorem 1. (Classical Congruence Extension Property) *Suppose we are given* $\triangle$**ABC** $\cong$ $\triangle$**DEF**. *Then* $\blacklozenge$**ABC** *is also congruent to* $\blacklozenge$**DEF**.

Proof. By the results of Section $\mathbf{II.4}$ there is a Galilean transformation **T** which sends **A, B** and **C** to **D, E** and **F** respectively; furthermore, this transformation sends $\triangle$**ABC** to $\triangle$**DEF**. We shall use the fact that **T** preserves barycentric coordinates to prove that **T** also sends $\blacklozenge$**ABC** to $\blacklozenge$**DEF**. Let **P** be a typical point in $\mathbf{R}^2$, and using barycentric coordinates expand **P** as a linear combination $x\mathbf{A} + y\mathbf{B} + z\mathbf{C}$, where $x + y + z = 1$. By definition, **P** lies in $\blacklozenge$**ABC** if and only if x, y, and z are all nonnegative. Since $\mathbf{T(P)} = x\mathbf{D} + y\mathbf{E} + z\mathbf{F}$, it follows that **T(P)** lies in $\blacklozenge$**DEF** if and only if the same condition is satisfied. Therefore **T** maps $\blacklozenge$**ABC** to $\blacklozenge$**DEF** as required. ■

We are going to need two additional undefined concepts to begin. The first is a family of ***plane measurable subsets*** $\mathcal{M}(\mathbf{R}^2)$, often abbreviated to $\mathcal{M}$, with the following simple properties:

Axiom PM – 1: *The family $\mathcal{M}$ contains all closed interiors of convex polygons.*

Axiom PM – 2: *The family $\mathcal{M}$ is closed under taking set – theoretic unions, intersections and differences.*

We shall be particularly interested in the bounded subsets of $\mathcal{M}$; namely, those that are contained in some square of the form $-a \leq x, y \leq a$ for some real number a.

The second undefined concept is an ***area function*** $\mathcal{a}$, which defines for each bounded subset **S** in $\mathcal{M}$ a nonnegative real number $\mathcal{a}$**(S)** called the ***area*** of **S**, and this function $\mathcal{a}$ is assumed to have the following properties:

Axiom PM – 3 (*Normalization condition*) **:** *The area of a rectangular region* ◆**ABCD** *whose sides have lengths p and q is equal to the product pq.*

This axiom can be weakened at the expense of some extra work, but clearly we need to know something about the area of at least one nontrivial figure in order to get started.

Axiom PM – 4 (*Areas of collinear sets*) **:** *The area of a bounded measurable subset of a line is equal to zero.*

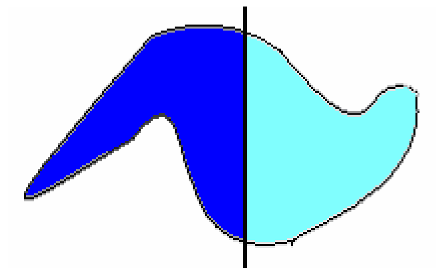
This is one way of ensuring that familiar one – dimensional subsets have zero areas.

Axiom PM – 5 (*Invriance under congruence*) **:** *If two bounded subsets of $\mathcal{M}$ are congruent, then their areas are equal.*

This principle is used frequently to help derive area formulas in elementary geometry.

Axiom PM – 6 (*Finite Additivity Postulate*) **:** *If a bounded set* **S** *in $\mathcal{M}$ is the union of two similar sets* $\mathbf{S_1}$ *and* $\mathbf{S_2}$, *and the sets* $\mathbf{S_1}$ *and* $\mathbf{S_2}$ *intersect in a set whose area is equal to zero, then the area of* **S** *is the sum of the areas of* $\mathbf{S_1}$ *and* $\mathbf{S_2}$.

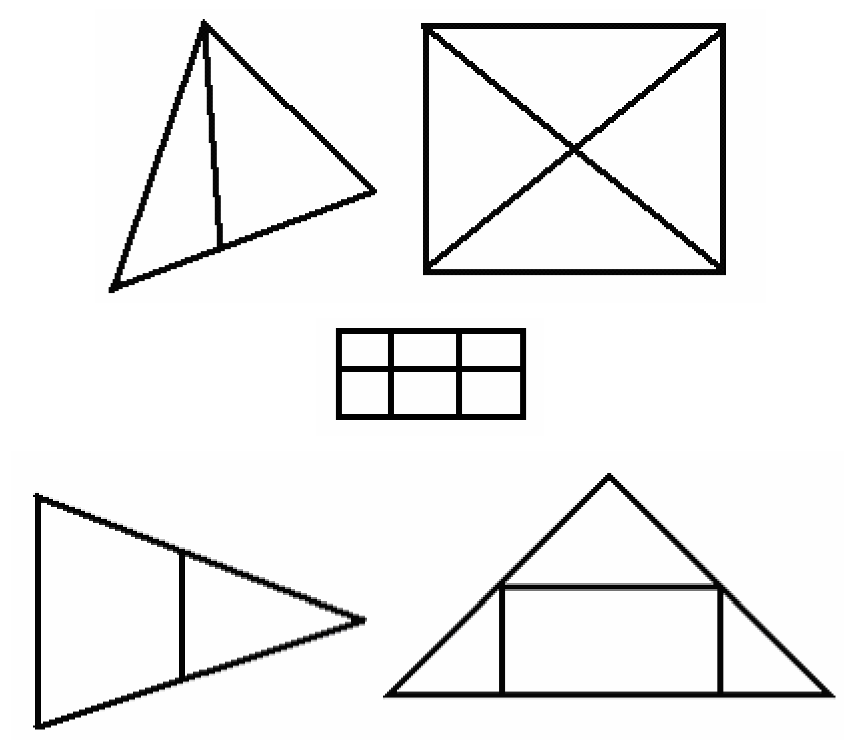
The final axiom is illustrated by the drawing below. Specifically, the area of the shaded region is given by adding the areas of the darker shaded region to the left and the lighter shaded region to the right. The intersection of these closed regions is contained in the vertical line in the middle, and this intersection has area equal to zero because it is contained in a line.

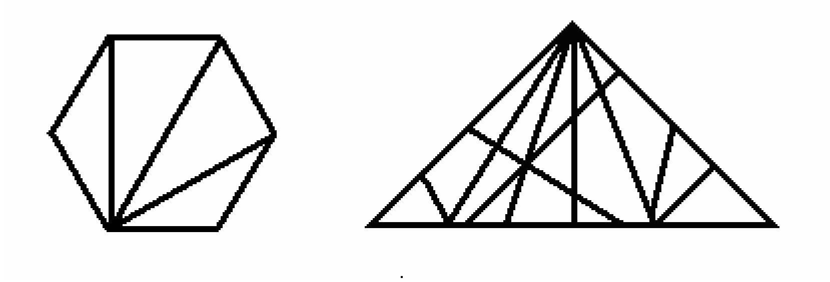


Our next goal is to explain why these axioms yield the usual for the areas of familiar objects. However, before we do so we need to digress and state some general properties of closed polygonal regions.

Decompositions of regular polygons

Classical derivations of area formulas for familiar plane figures often depend upon cutting a closed convex polygonal region up into smaller ***nonoverlapping*** regions of the same type. Specifically, we are generally given a closed convex polygonal region which can be decomposed into a finite union of smaller such regions associated to polygons $\mathbf{P}_m$ such that the intersection of any two is a common edge of two bounding polygons or a common vertex of two or more bounding polygons. Several examples are depicted below. Note in particular that many of the smaller polygons can share a given vertex and that one cannot draw any conclusions about the number of sides in the large polygon from the number of sides in the smaller polygons and vice versa.



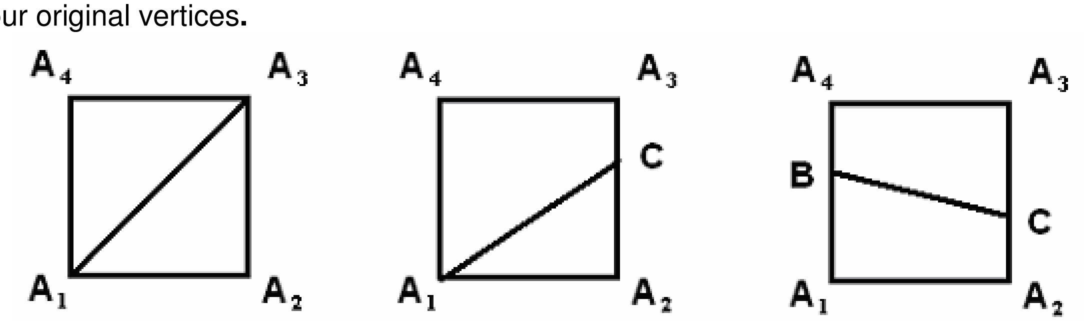


Of course, the simplest situations involve convex polygons that are split into two pieces by a line. The following result describes such situations.

<u>**Theorem 2.** (Folding and Cutting Principle)</u> *Suppose we are given a convex polygon of the form* $\mathbf{A}_1 \ldots \mathbf{A}_n$, *and suppose that* **B** *and* **C** *are points of* $\mathbf{A}_1 \ldots \mathbf{A}_n$ *which do not lie on the same edge of the polygon. Then the following hold*:

1. *If* **B** *is the vertex* $\mathbf{A}_1$ *and* **C** *is the vertex* $\mathbf{A}_m$ *where* m *is not equal to* 1 *or* n, *then the sets* $\{\mathbf{A}_1, \ldots, \mathbf{A}_m\}$ *and* $\{\mathbf{A}_m, \ldots, \mathbf{A}_n, \mathbf{A}_1\}$ *are the vertices of convex polygons such that* $\blacklozenge\mathbf{A}_1 \ldots \mathbf{A}_m \cup \blacklozenge\mathbf{A}_m \ldots \mathbf{A}_n\mathbf{A}_1 = \blacklozenge\mathbf{A}_1 \ldots \mathbf{A}_n$ *and* $\blacklozenge\mathbf{A}_1 \ldots \mathbf{A}_m \cap \blacklozenge\mathbf{A}_m \ldots \mathbf{A}_n\mathbf{A}_1 = [\mathbf{A}_1\mathbf{A}_m]$.
2. *If* **B** *is the vertex* $\mathbf{A}_1$ *and* **C** *lies on the open segment* $(\mathbf{A}_m\mathbf{A}_{m+1})$ *where* m *is not equal to* 1 *or* n, *then the sets* $\{\mathbf{A}_1, \ldots, \mathbf{A}_m, \mathbf{C}\}$ *and* $\{\mathbf{C}, \mathbf{A}_{m+1}, \ldots, \mathbf{A}_n, \mathbf{A}_1\}$ *are the vertices of convex polygons such that* $\blacklozenge\mathbf{A}_1 \ldots \mathbf{A}_m\mathbf{C} \cup \blacklozenge\mathbf{C}\mathbf{A}_{m+1} \ldots \mathbf{A}_n\mathbf{A}_1 = \blacklozenge\mathbf{A}_1 \ldots \mathbf{A}_n$ *and* $\blacklozenge\mathbf{A}_1 \ldots \mathbf{A}_m\mathbf{C} \cap \blacklozenge\mathbf{C}\mathbf{A}_{m+1} \ldots \mathbf{A}_n\mathbf{A}_1 = [\mathbf{A}_1\mathbf{C}]$.
3. *If* **B** *lies on the open segment* $(\mathbf{A}_n\mathbf{A}_1)$ *and* **C** *lies on another open segment of the form* $(\mathbf{A}_m\mathbf{A}_{m+1})$ *where* m *is not equal to* 1, *then the sets* $\{\mathbf{B}, \mathbf{A}_1, \ldots, \mathbf{A}_m, \mathbf{C}\}$ *and* $\{\mathbf{C}, \mathbf{A}_{m+1}, \ldots, \mathbf{A}_n, \mathbf{B}\}$ *are the vertices of convex polygons for which we have* $\blacklozenge\mathbf{B}\mathbf{A}_1 \ldots \mathbf{A}_m\mathbf{C} \cup \blacklozenge\mathbf{C}\mathbf{A}_{m+1} \ldots \mathbf{A}_n\mathbf{B} = \blacklozenge\mathbf{A}_1 \ldots \mathbf{A}_n$ *and also* $\blacklozenge\mathbf{B}\mathbf{A}_1 \ldots \mathbf{A}_m\mathbf{C} \cap \blacklozenge\mathbf{C}\mathbf{A}_{m+1} \ldots \mathbf{A}_n\mathbf{B} = [\mathbf{BC}]$.

The name "***folding and cutting***" is used because this construction corresponds to taking a closed polygonal region cut from a sheet of paper, folding it along some line which passes through the interior of the polygon, and cutting the polygonal region into two pieces along the fold. Here are drawings to illustrate the three cases when there are four original vertices.



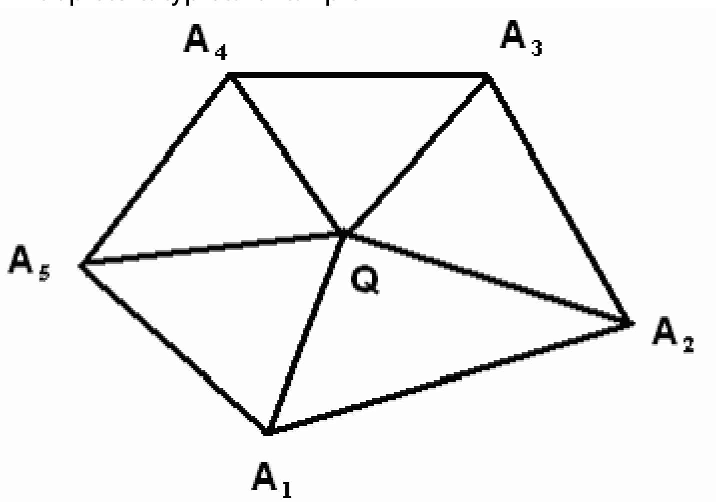
Note that <u>*we can obtain similar results if*</u> **B** <u>*is some vertex other than*</u> $\mathbf{A}_1$ <u>*or*</u> **B** <u>*lies on an open segment other than*</u> $(\mathbf{A}_n\mathbf{A}_1)$. For example, if we permute the roles of the $\mathbf{A}_k$

cyclically, with k going to $k + 1$ if $k < n$ and n going to 1, then we obtain similar conclusions if **B** $=$ **A**$_n$ or **B** lies on (**A**$_{n-1}$**A**$_n$), if we do this twice we get the same if **B** $=$ **A**$_{n-1}$ or **B** lies on (**A**$_{n-2}$**A**$_{n-1}$), and likewise if we do this more than twice.

The second result involves splitting a regular polygon into pieces using an interior point.

Theorem 3. (Star Decomposition Principle) *Suppose we are given a convex polygon of the form* **A**$_1$ **…** **A**$_n$, *and suppose that* **Q** *lies in the interior of* **A**$_1$ **…** **A**$_n$. *Then the closed polygonal region* ◆**A**$_1$ **…** **A**$_n$ *is the union of the regions* ◆**Q A**$_1$**A**$_2$, **…** , ◆**Q A**$_{n-1}$**A**$_n$, *and* ◆**Q A**$_n$**A**$_1$. *The intersection of two such closed regions is either a common edge of two triangles or the one point set* {**Q**}.

The drawing below depicts a typical example:

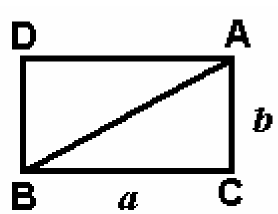


It is possible to prove these results with the techniques we have developed in this course, but the proofs are long and tedious, and since the arguments do not shed much light on the central questions of this section, the details will be omitted. ■

Derivations of some area formulas

Aside from the area formulas for rectangles, the next most basic examples are triangles. We begin with right triangles.

Theorem 4. *Suppose we have* △**ABC** *such that* |∠**ACB**| $=$ **90** *with* d(**A, C**) $=$ b *and* d(**B, C**) $=$ a. *If* ◆**ABC** *is the region bounded by* △**ABC**, *then* $\mathcal{a}$(◆**ABC**) $=$ ½ab.



Proof. Let **L** be the unique line through **A** that is parallel to **BC,** and let **M** be the unique line through **B** which is parallel to **AC.** Since lines perpendicular to intersecting lines will intersect, it follows that **L** meets **M** in some point **D;** it follows that **L** $=$ **AD** and

M = BD. Furthermore, by the parallelism and perpendicularity conditions we can also conclude that **L** is perpendicular to **AC** and **M** is perpendicular to **BC.** In particular, since **L** and **BC** are parallel and **BC** is perpendicular to **M,** it also follows that **L** is perpendicular to **M.** Since **L = AD** and **M = BD,** it follows that **AD** $\perp$ **BD.** Therefore we have shown that **B, C, A *and* D** (in that order) ***form the vertices of a rectangle*!!**

By the normalization axiom, the area of the closed polygonal region **S = ◆ACBD** bounded by the rectangle □**BCAD** is equal to ab. The first case of the Cutting and Folding Principle now implies that **◆ACBD = ◆ACB** $\cup$ **◆ADB** and **◆ACB** $\cap$ **◆ADB = [AB].** This means that

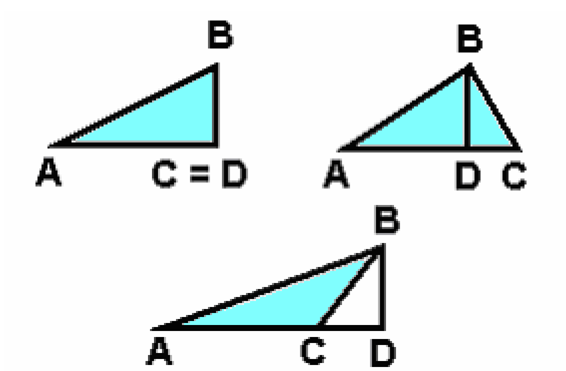
$$a(\blacklozenge \mathbf{ACBD}) \;=\; a(\blacklozenge \mathbf{ADB}) + a(\blacklozenge \mathbf{ACB}).$$

Furthermore, since the opposite sides of a rectangle have equal length we have d**(A, D) =** d**(B, C) =** a and d**(B, D) =** d**(A, C) =** b. Combining these with the assumption that $|\angle$**ACB**$|$ **=** $|\angle$**BDA**$|$ **= 90,** we conclude that $\triangle$**ACB** $\cong$ $\triangle$**BDA.** By the Classical Congruence Extension Property mentioned above, we also know that **◆ACB** is congruent to **◆BDA.** Thus we also have a**(◆ADB) =** a**(◆ACB).** Combining the two equations above, we obtain a**(◆ACBD) = 2**a**(◆ADB),** and since the left hand side is equal to ab it follows that **2**a**(◆ADB)** must be equal to ½ab. ■

The next step is to generalize the area formula to arbitrary triangles.

<u>Theorem 5.</u> Suppose we are given $\triangle$**BAC** and that **D** is the foot of the perpendicular from **B to AC.** Let d**(A, C) =** b and d**(B, D) =** h. Then a**(◆ABC) =** ½bh.

<u>Proof.</u> We must consider several cases depending upon how **D** is related to **A** and **C.** Specifically, the possibilities are **D*A*C, D = A, A*D*C, D = C,** and **A*C*D.** The first and fifth correspond to each other if we reverse the roles of **A** and **C,** and the first and fifth correspond to each other if we reverse the roles of **A** and **C,** so it suffices to consider the last three cases.



The case **C = D** is demonstrated in the previous theorem. Suppose now that **A*D*C.** By the second part of the Cutting and Folding Principle we have **◆ABC = ◆ADC** $\cup$ **◆BDC** and **◆ADC** $\cap$ **◆BDC = [BD].** This means that

$$a(\blacklozenge \mathbf{ABC}) \;=\; a(\blacklozenge \mathbf{ADC}) + a(\blacklozenge \mathbf{BDC}) =$$

$$\tfrac{1}{2}\, d(\mathbf{A, D})h + \tfrac{1}{2}\, d(\mathbf{D, C})h = \tfrac{1}{2}\,[\,d(\mathbf{A, D}) + d(\mathbf{D, C})\,]h.$$

Since **A∗D∗C** holds, the term inside the brackets is equal to d**(A, C)** and hence the area of ◆**ABC** is equal to ½ d**(A, C)**h. Finally, since the term on the right is equal to b, it follows that a(◆**ABC**) = ½bh.

Finally, suppose now that **A∗C∗D.** By the second part of the Cutting and Folding Principle we now have ◆**ADB** = ◆**ACB** ∪ ◆**CDB** and ◆**ACB** ∩ ◆**CDB** = **[CD].** This means that

$$a(◆\mathbf{ADB}) \;=\; a(◆\mathbf{ACB}) + a(◆\mathbf{CDB}) \;=\; a(◆\mathbf{ACB}) + \tfrac{1}{2}\, d(\mathbf{D, C})h$$

However, by the previous theorem we also have, and if we make this substitution we obtain the equation

$$\tfrac{1}{2}\, d(\mathbf{A, D})h \;=\; a(◆\mathbf{ADB}) \;=\; a(◆\mathbf{ACB}) + \tfrac{1}{2}\, d(\mathbf{D, C})h$$

which we may rewrite as

$$a(◆\mathbf{ACB}) = \tfrac{1}{2}\, d(\mathbf{A, D})h - \tfrac{1}{2}\, d(\mathbf{D, C})h = \tfrac{1}{2}\,[d(\mathbf{A, D}) - d(\mathbf{D, C})]h.$$

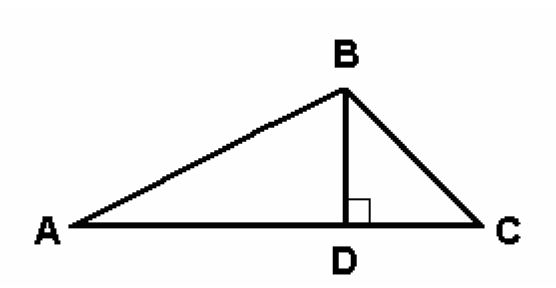
Since **A∗C∗D** holds, the term inside the brackets is equal to d**(A, C)** and hence the area of ◆**ABC** is equal to ½ d**(A, C)**h. Finally, since the term on the right is equal to b, it follows that a(◆**ABC**) = ½bh. This completes the verification of the area formula in all cases. ■

One can also find the area of ◆**ABC** in terms of the lengths of its sides using a formula named after Heron (or Hero) of Alexandria (10 A.D. – 75 A.D.).

<u>**Theorem 6.** (Heron's Formula)</u> *Given* △**ABC,** *denote the lengths of its three sides by* d**(B, C)** = **a**b, d**(A, C)** = b, *and* d**(A, B)** = c, *and let* s = ½($a + b + c$). *Then we have*

$$a(◆\mathbf{ABC}) \;=\; \mathbf{sqrt}\big(s(s-a)(s-b)(s-c)\big).$$

Proof. We know that at least two of the vertex angle measures for the triangle are less than **180,** and without loss of generality we might as well assume that both |∠**BCA**| and |∠**CAB**| are less than **90;** the other cases will follow by interchanging the roles of **A, B** and **C.** Let **D** ∈ **AC** be such that **BD** is perpendicular to **AC.** Then a corollary to the Exterior Angle Theorem implies that **D** lies on the open segment **(AC).**



Let d**(B, D)** = h, and let d**(A, D)** = x, so that d**(C, D)** = $b - x$. The central idea will be to solve for h in terms of s, a, b, and c using the Pythagorean Theorem and some algebraic manipulations.

Applying the Pythagorean Theorem to right triangles △**ADB** and △**BDC,** we obtain the equations

$$x^2 + h^2 = c^2 \qquad\qquad (b - x)^2 + h^2 = a^2$$

and if we solve for h^2 we obtain the equations $c^2 - x^2 = h^2 = a^2 - b^2 + 2bx - x^2$. Adding x^2 to each side of this equation yields $c^2 = a^2 - b^2 + 2bx$, and if we solve this for x we find that $x = (c^2 - a^2 + b^2)/2b$. Substituting this back into the first equation we find that

$$h^2 = c^2 - x^2 = c^2 - [(c^2 - a^2 + b^2)/2b]^2.$$

If Q denotes the area of ◆**ABC,** then we know that $Q = hb/2$, and therefore we have

$$Q^2 = h^2 b^2/4 = [4c^2b^2 - (c^2 - a^2 + b^2)^2]/16 =$$

$$(2a^2c^2 + 2a^2b^2 + 2c^2b^2 - a^4 - c^4 - b^4)/16.$$

The final expression for Q^2 should look promising because it is symmetric in a, b and c. .We must now see if we can to rewrite this expression more concisely. The key to doing so is the following algebraic identity, which may be checked directly by expanding the right hand side:

$$2a^2c^2 + 2a^2b^2 + 2c^2b^2 - a^4 - c^4 - b^4 =$$

$$(a + b + c)(-a + b + c)(a - b + c)(a + b - c)$$

If we let $p = a + b + c$ (the perimeter), then we have

$$(a + b + c)(-a + b + c)(a - b + c)(a + b - c) = p(p - 2a)(p - 2b)(p - 2c)$$

and we may use these equations to write to write

$$Q^2 = p(p - 2a)(p - 2b)(p - 2c)/16.$$

If we now let $p = 2s$, then the preceding equation becomes

$$Q^2 = p(p - 2a)(p - 2b)(p - 2c)/16 = 2s(2s - 2a)(2s - 2b)(2s - 2c)/16 =$$

$$16s(s - a)(s - b)(s - c)/16 = s(s - a)(s - b)(s - c)$$

and if we take square roots of both sides we obtain the area formula in the statement of the theorem. ■

Brahmagupta's_formula. A remarkable analog of Heron's Formula for cyclic convex quadrilaterals ($i.e.$, the vertices lie on a circle) was discovered by the prominent Hindu mathematician Brahmagupta (598 – 670). Specifically, suppose that **A, B, C, D** are the vertices of a convex quadrilateral which all lie on some circle Γ, denote the lengths of their sides by a, b, c, and d, and let s be equal t = $\frac{1}{2}(a + b + c + d)$. Then we have

$$\mathcal{a}(\blacklozenge \mathbf{ABCD}) = \mathrm{sqrt}\big(s(s - a)(s - b)(s - c)(s - d)\big).$$

Further information on proofs for this result may be found at the following online sites:

http://jwilson.coe.uga.edu/emt725/brahmagupta/brahmagupta.html

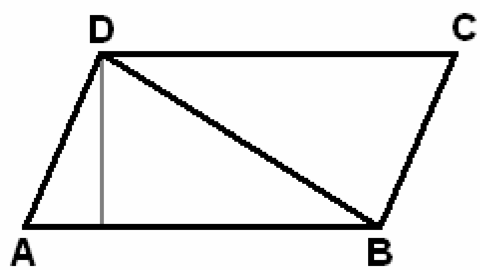
http://en.wikipedia.org/wiki/Brahmagupta's_formula

The next results are the area formulas for parallelograms and trapezoids. Recall from Section $\mathrm{III.3}$ that if **L** and **M** are parallel lines and **N** = **AB** is a line which is perpendicular to **L** and **M** at **A** and **B** respectively, then *the distance* d**(A, B)** *depends only on* **L** *and* **M**; in other words, if we are given any other $\mathbf{N}^*$, $\mathbf{A}^*$ and $\mathbf{B}^*$ with the corresponding properties, then $d(\mathbf{A}, \mathbf{B}) = d(\mathbf{A}^*, \mathbf{B}^*)$.

Theorem 7. *Suppose that* **A, B, C** *and* **D** *form the vertices of a convex quadrilateral (in that order) such that* **AB || CD.** *Let* h *denote the distance from a point on one of these two lines to the other line.*

1. *If* **BC || AD** *(so that the quadrilateral is a parallelogram) and* $b = d(\mathbf{A}, \mathbf{B})$, *then the area of* ◆**ABCD** *is equal to* bh.
2. *If* **BC** *and* **AD** *are not parallel (so that the quadrilateral is a proper trapezoid) such that we have* $b_1 = d(\mathbf{A}, \mathbf{B})$ *and* $b_2 = d(\mathbf{C}, \mathbf{D})$, *then the area of* ◆**ABCD** *is equal to* $\frac{1}{2}(b_1 + b_2)h$.

Proof. We first consider the case, in which the convex quadrilateral **ABCD** is assumed to be a ***parallelogram.***



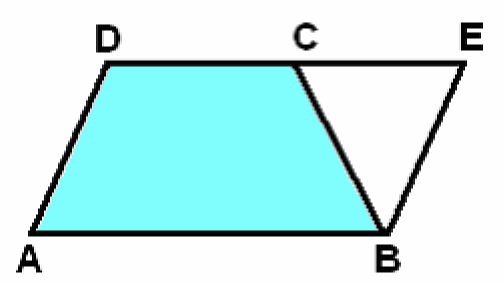
By the fundamental results on parallelograms, we have $d(\mathbf{A}, \mathbf{B}) = d(\mathbf{C}, \mathbf{D})$ and $d(\mathbf{A}, \mathbf{D}) = d(\mathbf{C},\mathbf{B})$. Since $d(\mathbf{B}, \mathbf{D}) = d(\mathbf{D}, \mathbf{B})$, it follows that $\triangle\mathbf{ABD} \cong \triangle\mathbf{CDB}$ by SSS. Therefore the Classical Congruence Extension Property and the invariance of area under congruence imply that a(◆**ABD**) = a(◆**CDB**).

By the first part of the folding and cutting principle, it follows that ◆**ABCD** = ◆**ABD** $\cup$ ◆**CDB** and ◆**ABD** $\cap$ ◆**CDB** = **[BD]**, so that

$$a(\blacklozenge\mathbf{ABCD}) = a(\blacklozenge\mathbf{ABD}) + a(\blacklozenge\mathbf{CDB}) = 2a(\blacklozenge\mathbf{ABD}).$$

Since a(◆**ABD**) = $\frac{1}{2}bh$, it follows that a(◆**ABCD**) = bh, as required in the case of parallelograms.

Suppose now that the quadrilateral is a ***trapezoid*** with **AB || CD.** If $d(\mathbf{A}, \mathbf{B}) = d(\mathbf{C}, \mathbf{D})$ then the quadrilateral is a parallelogram, so we might as well assume that the lengths of the opposite parallel sides are unequal. Strictly speaking, there are two cases depending upon whether $d(\mathbf{A}, \mathbf{B}) > d(\mathbf{C}, \mathbf{D})$ or $d(\mathbf{A}, \mathbf{B}) < d(\mathbf{C}, \mathbf{D})$, but as usual we can extract the second case from the first by interchanging the roles of **A** and **B** with those of **C** and **D** respectively. As in the statement of the theorem, let $b_1 = d(\mathbf{A}, \mathbf{B})$ and $b_2 = d(\mathbf{C}, \mathbf{D})$.



Let **E** = **D + B – A,** so that **A, B, E** and **D** (in order) are the vertices of a parallelogram. Since both **DE** and **CD** are parallel to **AB,** it follows that **DC** = **DE.** We claim that **D*C*E** holds. Since **CD || AB,** it follows that **C** = **D** + k**(B – A)** for some scalar k**.** Furthermore, since we know that **A, B, C** and **D** form the vertices of a trapezoid it follows that **C** lies in the interior of ∠ **DAB;** in terms of barycentric coordinates, this means that k must be positive. Therefore we have

$$b_2 \;=\; d(\mathbf{C},\mathbf{D}) \;=\; \|\mathbf{C}-\mathbf{D}\| \;=\; k\|\mathbf{B}-\mathbf{A}\| \;=\; kb_1$$

and since $b_2 < b_1$ it follows that $k < 1$. Therefore we have the desired order relation **B*C*E.**

By the second part of the folding and cutting principle, we have ◆**ABCE** = ◆**ABCD** ∪ ◆**BCE** and ◆**ABCD** ∩ ◆**BCE** = **[BC].** Therefore we also have the equation

$$a(\blacklozenge\mathbf{ABED}) \;=\; a(\blacklozenge\mathbf{ABCD}) + a(\blacklozenge\mathbf{BCE}).$$

By the previously established formula for parallelograms we know that $a(\blacklozenge\mathbf{ABED}) = b_1h$ and $a(\blacklozenge\mathbf{BCE}) = \tfrac{1}{2}(b_1 - b_2)h$. It follows that

$$a(\blacklozenge\mathbf{ABCD}) \;=\; a(\blacklozenge\mathbf{ABED}) - a(\blacklozenge\mathbf{BCE}) \;=\;$$

$$b_1h - \tfrac{1}{2}(b_1-b_2)h \;=\; (b_1 - \tfrac{1}{2}b_1 + \tfrac{1}{2}b_2)h \;=\; \tfrac{1}{2}(b_1+b_2)h$$

which is the formula stated in the theorem.■

The final area formula in this section will cover ***regular polygons***. In order to state this formula, we shall need some definitions. The ***perimeter*** of an arbitrary convex polygon $\mathbf{A}_1 \ldots \mathbf{A}_n$ is defined as usual to be the sum of the lengths of the sides:

$$p \;=\; d(\mathbf{A}_1,\mathbf{A}_2) + \ldots + d(\mathbf{A}_{n-1},\mathbf{A}_n) + d(\mathbf{A}_n,\mathbf{A}_1)$$

Proposition 8. *Suppose that* $\mathbf{A}_1 \ldots \mathbf{A}_n$ *is a convex polygon, and let* **Q** *be its center. For each* $k = 1, \ldots, n$ *let* $\mathbf{C}_k$ *be the foot of the perpendicular from* **Q** *to* $\mathbf{A}_k\mathbf{A}_{k+1}$ *(here* $\mathbf{A}_{n+1} = \mathbf{A}_1$ *by our usual numbering conventions). Then all of the distances* $d(\mathbf{Q},\mathbf{C}_k)$ *are equal.*

This common value is called the ***apothem*** (pronounced "**AP** – *o* – **them**" with all short vowels, the "**th**" as in "**thin**," the heaviest accent on the first syllable, and a secondary accent on the last syllable).

Proof. By the description of regular polygons in Section **III.3,** we know that

$$d(\mathbf{A}_1,\mathbf{Q}) \;=\; d(\mathbf{A}_2,\mathbf{Q}) \;=\; \ldots \;=\; d(\mathbf{A}_n,\mathbf{Q})$$

and we also know that

$$d(\mathbf{A}_1, \mathbf{A}_2) \;=\; \dots \;=\; d(\mathbf{A}_{n-1}, \mathbf{A}_n) \;=\; d(\mathbf{A}_n, \mathbf{A}_1),$$

so that SSS implies

$$\triangle \mathbf{QA}_1\mathbf{A}_2 \;\cong\; \dots \;\cong\; \triangle \mathbf{QA}_{n-1}\mathbf{A}_n \;\cong\; \triangle \mathbf{QA}_n\mathbf{A}_1 .$$

Furthermore, all these triangles are isosceles triangles; therefore, if $\triangle \mathbf{QA}_k\mathbf{A}_{k+1}$ is one of these triangles (with the standard convention if $k = n$) and $\mathbf{C}_k$ is the foot of the perpendicular from **Q** to $\mathbf{A}_k\mathbf{A}_{k+1}$, then $\mathbf{C}_k$ lies on the open segment $(\mathbf{A}_k\mathbf{A}_{k+1})$ and in fact is its midpoint (by the characterization of perpendicular bisectors). Therefore we have $d(\mathbf{A}_k, \mathbf{C}_k) = \frac{1}{2}\, d(\mathbf{A}_k, \mathbf{A}_{k+1})$ for all **k,** and hence we also have

$$d(\mathbf{A}_1, \mathbf{C}_1) \;=\; \dots \;=\; d(\mathbf{A}_{n-1}, \mathbf{C}_{n-1}) \;=\; d(\mathbf{A}_n, \mathbf{C}_n).$$

By SSS we now have

$$\triangle \mathbf{QA}_1\mathbf{C}_1 \;\cong\; \dots \;\cong\; \triangle \mathbf{QA}_{n-1}\mathbf{C}_{n-1} \;\cong\; \triangle \mathbf{QA}_n\mathbf{C}_n$$

and the latter implies the desired string of equations $d(\mathbf{Q}, \mathbf{C}_1) = \dots = d(\mathbf{Q}, \mathbf{C}_{n-1}) = d(\mathbf{Q}, \mathbf{C}_n)$. ■

Before turning to the area formula for regular polygons, we shall dispose of one step in the proof that is valid for an arbitrary convex polygon.

Proposition 9. *Suppose that* $\mathbf{A}_1, \dots, \mathbf{A}_n$ *are the vertices of a convex polygon (taken in that order), and let* **Q** *be a point in the interior of this polygon. Then we have*

$$\mathcal{a}(\blacklozenge \mathbf{A}_1 \dots \mathbf{A}_n) \;=\; \mathcal{a}(\blacklozenge \mathbf{QA}_1\mathbf{A}_2) + \dots + \mathcal{a}(\blacklozenge \mathbf{QA}_{n-1}\mathbf{A}_n) \;+\; \mathcal{a}(\blacklozenge \mathbf{Q}\,\mathbf{A}_n\mathbf{A}_1).$$

Proof. For $k = 2, \dots, n$ let $\mathbf{X}_k$ be the set $\blacklozenge \mathbf{QA}_1\mathbf{A}_2 \cup \dots \cup \blacklozenge \mathbf{QA}_{k-1}\mathbf{A}_k$, so that the Star Decomposition Property implies $\mathbf{X}_k = \mathbf{X}_{k-1} \cup \blacklozenge \mathbf{QA}_{k-1}\mathbf{A}_k$ and $\mathbf{X}_{k-1} \cap \blacklozenge \mathbf{QA}_{k-1}\mathbf{A}_k = [\mathbf{Q}\,\mathbf{A}_{k-1}]$. By the additivity property we then have the recursive equations

$$\mathcal{a}(\blacklozenge \mathbf{X}_k) \;=\; \mathcal{a}(\blacklozenge \mathbf{X}_{k-1}) + \mathcal{a}(\blacklozenge \mathbf{QA}_{k-1}\mathbf{A}_k)$$

and hence we have

$$\mathcal{a}(\blacklozenge \mathbf{X}_k) \;=\; \mathcal{a}(\blacklozenge \mathbf{Q}\,\mathbf{A}_1\mathbf{A}_2) + \dots + \mathcal{a}(\blacklozenge \mathbf{QA}_{k-1}\mathbf{A}_k)$$

for $k = 3, \dots, n$. Finally, we also have $\blacklozenge \mathbf{A}_1 \dots \mathbf{A}_n = \mathbf{X}_n \cup \blacklozenge \mathbf{QA}_n\mathbf{A}_1$ and $\mathbf{X}_n \cap \blacklozenge \mathbf{QA}_n\mathbf{A}_1 = [\mathbf{QA}_n] \cup [\mathbf{QA}_1]$. Now the two closed segments on the right hand side have areas equal to zero, and they only have the endpoint **Q** in common, and hence by the Additivity Property we know that the area of $[\mathbf{QA}_n] \cup [\mathbf{QA}_1]$ is also equal to zero. We can then apply the Additivity Property one more time to conclude that

$$\mathcal{a}(\blacklozenge \mathbf{A}_1 \dots \mathbf{A}_n) \;=\; \mathcal{a}(\blacklozenge \mathbf{X}_n) + \mathcal{a}(\blacklozenge \mathbf{Q}\,\mathbf{A}_n\mathbf{A}_1)$$

and if we combine this with the previous equation and the Star Decomposition Property we obtain

$$\mathcal{a}(\blacklozenge \mathbf{A}_1 \dots \mathbf{A}_n) \;=\; \mathcal{a}(\blacklozenge \mathbf{QA}_1\mathbf{A}_2) + \dots + \mathcal{a}(\blacklozenge \mathbf{QA}_{n-1}\mathbf{A}_n) + \mathcal{a}(\blacklozenge \mathbf{Q}\,\mathbf{A}_n\mathbf{A}_1)$$

which is the formula in the statement of the theorem. ■

Theorem 10. *Suppose that* $A_1, \dots, A_n$ *are the vertices of a regular polygon (taken in that order). If* p *denotes the perimeter of this regular polygon and* a *denotes its apothem, then* $\mathcal{a}(\blacklozenge A_1 \dots A_n) = \frac{1}{2}pa.$

Proof. Let **Q** be the center of the regular polygon. If we now apply the previous result and the area formula for triangles, we find that

$$\mathcal{a}(\blacklozenge A_1 \dots A_n) = \mathcal{a}(\blacklozenge QA_1A_2) + \dots + \mathcal{a}(\blacklozenge QA_{n-1}A_n) + \mathcal{a}(\blacklozenge Q\,A_nA_1) =$$

$$\tfrac{1}{2}d(A_1, A_2)\,a + \dots + \tfrac{1}{2}d(A_{n-1}, A_n)\,a + \tfrac{1}{2}d(A_n, A_1)a =$$

$$\tfrac{1}{2}\big[d(A_1, A_2) + \dots + d(A_{n-1}, A_n) + d(A_n, A_1)\big]a = \tfrac{1}{2}pa$$

which is the formula stated in the theorem. ■

We have only discussed some of the material about areas from elementary geometry. Further information can be found in Chapters 13 – 14 of the book by Moïse and the following online sites:

http://en.wikipedia.org/wiki/Area_(geometry)

http://www.gomath.com/htdocs/ToGoSheet/Geometry/area.html

Axioms for volumes

We are not going to prove any theorems about volumes of figures in space, but we shall state the axioms and mention some complications that arise when passing from two to three dimensions. Since the final axiom for volumes involves plane areas, it will also be necessary to discuss the role of plane area in three dimensions.

As in the planar case, the first thing we need is an undefined concept given by a family of ***measurable subsets*** $\mathcal{M}(\mathbf{R}^3)$, often abbreviated to $\mathcal{M}$, which is assumed to satisfy the following simple properties:

Axiom SM – 1: *The family* $\mathcal{M}$ *contains all of the* ***standard rectangular solids*** **S** = $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ *given by all points whose coordinates* (x, y, z) *satisfy the inequalities* $a_1 \leq x \leq b_1$, $a_2 \leq y \leq b_2$, *and* $a_3 \leq z \leq b_3$.

Given a plane **P** in space and a point **X** not on **P**, the ***closed half space*** $\mathbf{H}^{\#}(\mathbf{P}; \mathbf{X})$ is defined to be the union of the **P** with the side of **P** (in space) containing **X**.

Axiom SM – 2 : *If* **P** *is a plane in space and* **S** *belongs to* $\mathcal{M}$, *then the intersection* $\mathbf{S} \cap \mathbf{P}$ *belongs to* $\mathcal{M}$, *and if* **X** *is a point in space which does not lie on* **P**, *then the intersection* $\mathbf{S} \cap \mathbf{H}^{\#}(\mathbf{P}; \mathbf{X})$ *belongs to* $\mathcal{M}$. *Furthermore, the family* $\mathcal{M}_{\mathbf{P}}$ *of all subsets of* **P** *which lie in* $\mathcal{M}$ *satisfies the previous axioms* **AM – 1** *and* **AM – 2.**

Axiom SM – 3: *The family* $\mathcal{M}$ *is closed under taking set – theoretic unions, intersections and differences.*

We shall be particularly interested in the bounded subsets of $\mathcal{M}$; namely, those that are contained in some square of the form $-k \leq x, y, z \leq k$ for some real number k.

The next assumption states that planes in space have decent notions of area.

<u>**Axiom SM – 4:**</u> *For each plane* **P** *there is an area function* $a_{\mathbf{P}}$, *which is defined on the bounded subsets of the family* $\mathcal{M}_{\mathbf{P}}$ *and satisfies the previous axioms* **AM – 3** *through* **AM – 6.**

This axiom implicitly contains a second undefined concept; namely, a family of area functions $a_{\mathbf{P}}$, one for each plane **P**; frequently the subscript is omitted to in order to simplify the notation. The third undefined concept will be a ***volume function*** $\mathcal{V}$, which defines for each bounded subset **S** in $\mathcal{M}$ a nonnegative real number $\mathcal{V}$**(S)** called the <u>***volume***</u> of **S**, and this function $\mathcal{V}$ is assumed to have the following properties:

<u>**Axiom SM – 5** (*Normalization condition*) **:**</u> *The volume of the rectangular solid described in Axiom* **SM – 1** *is the product of the length of the sides; in other words, it is equal to* $(b_1 - a_1)(b_2 - a_2)(b_3 - a_3)$.

<u>**Axiom SM – 6** (*Areas of coplanar sets*) **:**</u> *The volume of a bounded measurable subset of a plane is equal to zero.*

<u>**Axiom SM – 7** (*Invariance under congruence*) **:**</u> *If two bounded subsets of* $\mathcal{M}$ *are congruent, then their volumes are equal.*

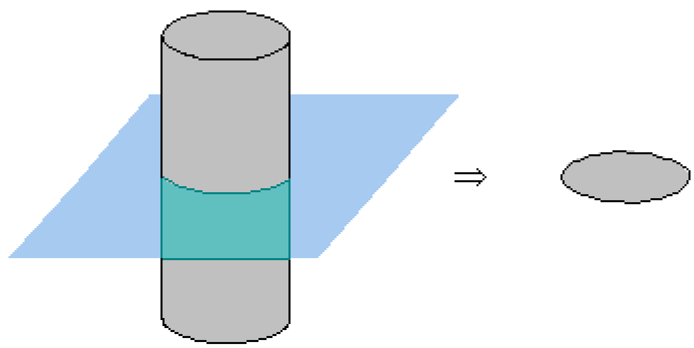
<u>**Axiom SM – 8** (*Finite Additivity Postulate*) **:**</u> *If a bounded set* **S** *in* $\mathcal{M}$ *is the union of two similar sets* $\mathbf{S_1}$ *and* $\mathbf{S_2}$, *and the sets* $\mathbf{S_1}$ *and* $\mathbf{S_2}$ *intersect in a set whose volume is equal to zero, then the volume of* **S** *is the sum of the volumes of* $\mathbf{S_1}$ *and* $\mathbf{S_2}$.

Each of the preceding four axioms is a direct analog of an axiom for plane area. However, we also need one further assumption:

<u>**Axiom SM – 9** (*Cavalieri's Principle*) **:**</u> *Suppose that we are given two bounded measurable sets* $\mathbf{S_1}$ *and* $\mathbf{S_2}$ *and also a plane* **P**, *and suppose further that for every plane* **Q** *parallel to* **P** *the intersections* $\mathbf{Q} \cap \mathbf{S_1}$ *and* $\mathbf{Q} \cap \mathbf{S_2}$ *have equal areas. Then* $\mathbf{S_1}$ *and* $\mathbf{S_2}$ *have equal volumes.*

This is clearly different from any of the plane area postulates, so we shall try to **(1)** make it plausible and **(2)** explain why it is needed.

As its name suggests, Axiom **SM – 9** reflects ideas advanced by B. Cavalieri (1598 – 1647); in fact, the key ideas were implicit in a work of Archimedes (287 B.C.E. – 212 B.C.E.) called <u>***The Method***</u>, but he viewed it as a tool to discovering new facts rather than a genuine mathematical result, and this work was lost and essentially unknown for many centuries until its rediscovery in 1909. Historically the principle represents one step in the development of integral calculus, and it can be explained fairly simply in such terms. In order to simplify the discussion, we shall assume that the plane **P** in Axiom **SM – 9** is the xy – plane (in any case this can be achieved by changing coordinates). A bounded measurable set **S** is then contained between two planes parallel to **P;** for the sake of definiteness we shall assume these planes are defined by $z = c$ and $z = d$ respectively, where $c < d$. For each t such that $c \leq t \leq d$, let $\mathbf{P}_t$ be the plane defined by $z = t$; then each set $\mathbf{P}_t \cap \mathbf{S}$ is a bounded measurable subset of the plane, and as such has an area $\mathbf{a}(t) = \mathcal{A}(\mathbf{P}_t \cap \mathbf{S})$; if $\mathbf{P}_t \cap \mathbf{S}$ is empty then by convention $\mathcal{A}(\mathbf{P}_t \cap \mathbf{S}) = 0$. An example is depicted on the next page in which **S** is a cylindrical region in space whose axis is perpendicular to **P,** and we also assume it is the piece whose upper and lower boundaries are the parallel planes $z = 0$ and $z = 1$. In this special case each of the slices $\mathbf{P}_t \cap \mathbf{S}$ is a closed disk (a circle together with its interior points), and the areas $\mathbf{a}(t)$ of the slices are equal to some fixed value B.



(**Source:** http://www.mathleague.com/help/geometry/3space.htm)

For the particular cylindrical example in the illustration, we know that the volume is equal to the product of B with $d - c = 1$. More generally, a "disk method" argument as in ordinary integral calculus will strongly suggest that in more general cases, where the areas $\mathbf{a}(t)$ of the slices $\mathbf{P}_t \cap \mathbf{S}$ may vary with t, then the volume of the solid should be given by the following integral formula:

$$\mathbf{V} = \int_0^1 \mathbf{a}(t)\, dt$$

Furthermore, if **S** is contained between two arbitrary parallel planes $z = c$ and $z = d$ where $c < d$, then one obtains a similar integral in which the lower and upper limits of integration are $c < d$ respectively. Suppose now that we have $\mathbf{S}_1$ and $\mathbf{S}_2$ satisfying the hypotheses of $\mathrm{SM} - 9$, and define

$$\mathbf{a}_k(t) = \mathcal{A}(\mathbf{P}_t \cap \mathbf{S}_k), \qquad k = 1, 2$$

so that the hypotheses of $\mathrm{SM} - 9$ yield $\mathbf{a}_1(t) = \mathbf{a}_2(t)$. The preceding discussion suggests that $\mathcal{V}(\mathbf{S}_k)$ is equal to the definite integral of $\mathbf{a}_k(t)$ from $z = c$ to $z = d$, and one can assume that the limits of integration are the same in both instances. Therefore the condition $\mathbf{a}_1(t) = \mathbf{a}_2(t)$ implies that $\mathcal{V}(\mathbf{S}_1)$ and $\mathcal{V}(\mathbf{S}_2)$ are integrals of the same function and hence should be equal. But this means that $\mathcal{V}(\mathbf{S}_1)$ and $\mathcal{V}(\mathbf{S}_2)$ should be equal. In fact, _one can justify all of this rigorously using machinery developed in_ *graduate level courses on measure theory*. We shall say more about the latter and its ties to elementary geometry at the end of this section.

The assumption of something like Cavalieri's Principle is not merely a matter of convenience, but on the contrary it is **logically unavoidable**. Early evidence for this appears in classical Greek geometry writings on geometry, where proofs of basic theorems on volumes are often far more complicated and delicate than the proofs for theorems on plane areas. There are numerous examples of this in the ***Elements***, and the subsequent work of Archimedes took things much further; much of this work was based upon a ***method of exhaustion***, which anticipated the use of limits but avoided doing so explicitly by means of very complicated proofs by contradiction. When the logical foundations of classical geometry were scrutinized near the end of the 19^{th} century, there was renewed interest in questions concerning the need for ideas from integral calculus, and during the first few years of the 20^{th} century M. Dehn (1878 – 1952) proved results confirming the need for some input related to limits and calculus in any mathematically complete treatment of even the most basic volume problems in elementary geometry (for example, finding the volumes of pyramids with triangular bases). There is some general information on Dehn's results and related topics in the online reference listed below:

http://en.wikipedia.org/wiki/Hilbert's_third_problem

Logical redundancy of area and volume axioms

We have already noted that many advanced treatments of elementary geometry do not discuss axioms for area and volume. One important reason is that mathematicians can define ***measurable sets***, ***areas*** and ***volumes*** for $\mathbf{R}^2$ and $\mathbf{R}^3$ in a unique way such that the basic properties in the axioms are satisfied; this is generally done using the theory of ***Lebesgue measure and integration*** named after H. Lebesgue (1875 – 1941). Virtually every graduate level text on real analysis or measure theory will provide highly detailed information on this subject. Here are some online references which summarize the main points of the theory:

http://en.wikipedia.org/wiki/Lebesgue_measure

http://mathworld.wolfram.com/LebesgueMeasure.html

There is also a considerably simpler theory of ***Jordan measure*** which is closely related to the theory of the Riemann integral in undergraduate real analysis courses and is adequate for the purposes of elementary geometry (but not for certain other classes of mathematical problems); this theory was developed somewhat earlier by C. Jordan (1838 – 1922). Some online references for Jordan measure are listed below:

http://en.wikipedia.org/wiki/Jordan_measure

http://mathworld.wolfram.com/JordanMeasure.html

In particular, we note that suitable versions of Cavalieri's Principle can be deduced as theorems in either of these measure theories. These are immediate consequences of ***Tonelli's Theorem*** or ***Fubini's Theorem***; these results of L. Tonelli (1885 – 1946) and G. Fubini (1879 – 1943) are extremely general versions of the advanced calculus principle for evaluating multiple integrals as iterated integrals.