CONVEX POLYGONAL REGIONS AND COORDINATE GEOMETRY

NOTE. *The first subheading below should be inserted after the discussion of the interior of a convex polygon on page 102 of the file* `geometrynotes3a.pdf`*, and the second subheading should be inserted after the definition of closed convex polygonal regions page 107 of the file* `geometrynotes3c.pdf`*.*

When the interior of a triangle was defined in Section II.3, a description of this region in terms of vector geometry (specifically, using barycentric coordinates) was given. A corresponding definition of interiors for convex polygons appears near the beginning of Section III.3, and a related notion of *closed polygonal region* plays an important role in Section III.7. Descriptions for both types of sets in terms of coordinate geometry are given below.

Coordinate geometry and interiors of convex polygons

Suppose that $n \geq 3$ and $\mathbf{a}_1, \cdots, \mathbf{a}_n$ (in the given order) are the vertices for a convex n-gon in $\mathbf{R}^2$; we shall adopt the cyclic numbering conventions of Section III.3 to define $\mathbf{a}_k$ for other integral values of k, so that $\mathbf{a}_k = \mathbf{a}_{k+n}$ for all k. For each k there is a linear equation $\langle \mathbf{u}_i, \mathbf{x} \rangle = b_i$ which defines the line $\mathbf{a}_{i-1}\mathbf{a}_i$., and by the definition of a convex polygon we know that the $(n-2)$ numbers

$$\langle \mathbf{u}_i, \mathbf{a}_j \rangle - b_i \qquad (j = i+1, \cdots, i+n-2)$$

all have the same sign. Replacing $\mathbf{u}_i$ and b_i with their negatives if necessary, we may assume that this sign is always positive. Thus the interior of the convex polygon $\mathbf{a}_1 \cdots \mathbf{a}_n$ is defined by a finite set of strict linear inequalities for which the displayed expressions are all positive.

Coordinate geometry and closed polygonal regions

Suppose we have a convex polygon as above and its interior is defined by the linear inequalities as chosen in the preceding discussion. In Section III.7 we defined the closed polygonal region $\bullet\, \mathbf{a}_1 \cdots \mathbf{a}_n$ to be the intersection of the closed half-planes $\mathbf{H}^{\#}(\mathbf{a}_{i-1}\mathbf{a}_i; \mathbf{a}_{i+1})$ for $1 \leq i \leq n$. In our setting, the latter are the sets defined by the inequalities

$$\langle \mathbf{u}_i, \mathbf{a}_j \rangle \geq b_i \qquad (j = i+1, \cdots i+n-2)$$

and it is also described by the following result:

CLAIM. *If $\mathbf{a}_1, \cdots, \mathbf{a}_n$ (in the given order) are as above, then $\bullet\, \mathbf{a}_1 \cdots \mathbf{a}_n$ is equal to the union of the convex polygon $\mathbf{a}_1 \cdots \mathbf{a}_n$ and its interior.*

Proof. If a half plane is defined by a strict inequality, then the associated closed half plane is defined as the set of all points which either satisfy the strict inequality or the related equation obtained by replacing the inequality sign with an equals sign. Therefore it follows immediately that every point in the interior of $\mathbf{a}_1 \cdots \mathbf{a}_n$ lies in $\bullet\ \mathbf{a}_1 \cdots \mathbf{a}_n$. Furthermore, we claim that every vertex lis in the latter set. To see this, let $\mathbf{a}_j$ be a vertex. Then by the definition of convex polygon we know that

$$\mathbf{a}_j \in \mathbf{H}^{\#}(\mathbf{a}_{i-1}\mathbf{a}_i; \mathbf{a}_{i+1})$$

if $i = j+2, j, \cdots, j+n-1$ and also

$$\mathbf{a}_j \in \mathbf{a}_{j-1}\mathbf{a}_j \cap \mathbf{a}_j\mathbf{a}_{j+1} \subset \mathbf{H}^{\#}(\mathbf{a}_{j-1}\mathbf{a}_j; \mathbf{a}_{j+1}) \cap \mathbf{H}^{\#}(\mathbf{a}_j\mathbf{a}_{j+1}; \mathbf{a}_{j+2})$$

so that $\mathbf{a}_j$ must belong to each of the intersecting sets which appear in the definition of $\bullet\ \mathbf{a}_1 \cdots \mathbf{a}_n$. Finally, since the latter is a convex set, it also follows that the entire closed segment $[\mathbf{a}_j\mathbf{a}_{j+1}]$ must be contained in $\bullet\ \mathbf{a}_1 \cdots \mathbf{a}_n$, completing the proof that the latter contains the original convex polygon.

We must now show that every point of $\bullet\ \mathbf{a}_1 \cdots \mathbf{a}_n$ must either lie on the convex polygon or its interior. Suppose we are given a point of $\bullet\ \mathbf{a}_1 \cdots \mathbf{a}_n$ which does not lie in the interior. Then it follows that the point $\mathbf{x}$ must lie on one of the lines $\mathbf{a}_j\mathbf{a}_{j+1}$, and it will suffice to show that $\mathbf{x}$ must lie on the closed segment $[\mathbf{a}_j\mathbf{a}_{j+1}]$. — Let us suppose that $\mathbf{x}$ lies on the line but not on the closed segment. Then the basic resutls on betweenness imply that either $\mathbf{a}_j * \mathbf{a}_{j+1} * \mathbf{x}$ or else $\mathbf{x} * \mathbf{a}_j * \mathbf{a}_{j+1}$. In the first case it would follow that $\mathbf{x}$ and $\mathbf{a}_j$ would lie on opposite sides of the line $\mathbf{a}_{j+1}\mathbf{a}_{j+2}$, which contradicts the definition of $\bullet\ \mathbf{a}_1 \cdots \mathbf{a}_n$. Likewise, in the second case it would follow that $\mathbf{x}$ and $\mathbf{a}_{j+1}$ would lie on opposite sides of the line $\mathbf{a}_{j-1}\mathbf{a}_j$, which also contradicts the definition of $\bullet\ \mathbf{a}_1 \cdots \mathbf{a}_n$. Thus the only points of the latter which lie on the line $\mathbf{a}_j\mathbf{a}_{j+1}$ are the points which lie on the closed segment joining these two points, and this completes the proof of the Claim.■