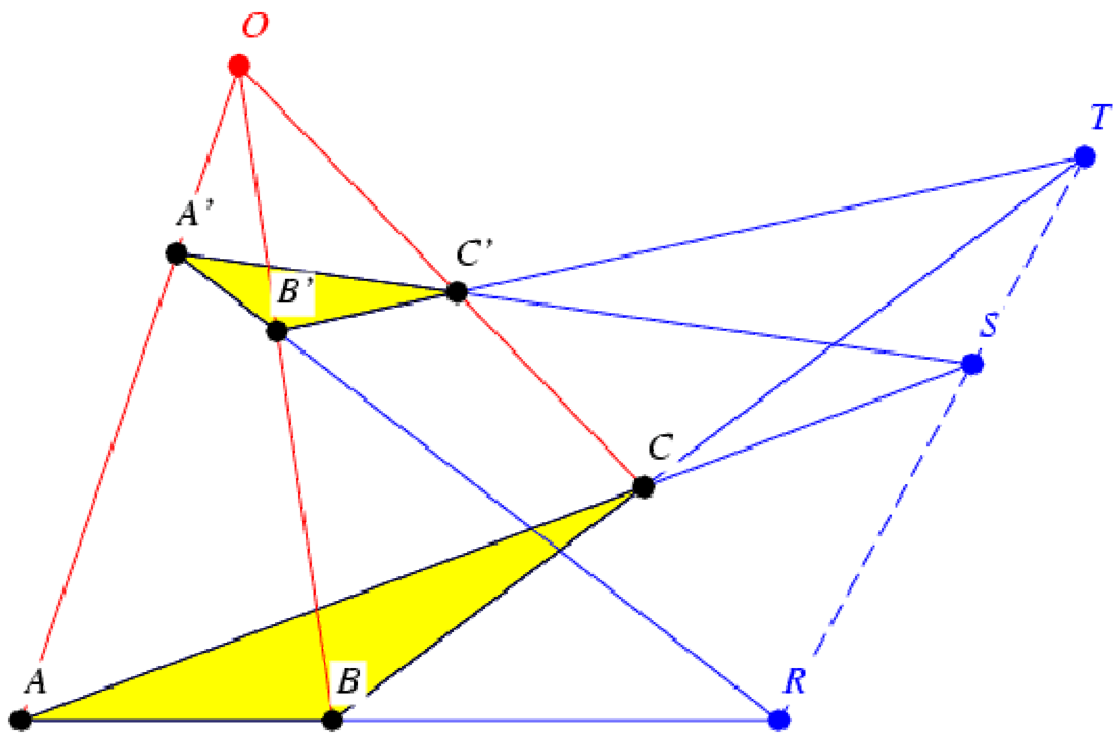
IV.5 : Theorems of Desargues and Pappus

The elegance of the[se] statements testifies to the unifying power of projective geometry. … The elegance of the[ir] proofs … testifies to the power of the method of homogeneous coordinates.

Ryan, p. 126

We have already mentioned Desargues' Theorem as an example of a result which is best understood by means of perspective projections and hence of projective geometry, and we shall begin by proving the nonplanar case of that result using the machinery that was developed in the preceding three sections.

Theorem 1. (Desargues' Theorem — nonplanar case). *Suppose that we are given four* ***noncoplanar*** *points* **Q, A, B, C** $\in \mathbb{P}(\mathbb{R}^3)$, *and suppose we are given three other points* **A′, B′** *and* **C′** *such that* **A′** $\in$ **QA, B′** $\in$ **QB,** *and* **C′** $\in$ **QC.** *Then the pairs of corresponding lines* **{ AB, A′B′ }, { AC, A′C′ },** *and* **{ BC, B′C′ }** *meet at points* **D, E** *and* **F** *respectively, and these three points are collinear.*



(Source: http://mathworld.wolfram.com/DesarguesTheorem.html)

In the picture above, the point **O** corresponds to the point **Q** in the statement of the theorem, and similarly the intersection points **R, S** and **T** correspond to **D, E** and **F** respectively.

Proof. The argument is very similar to the informal discussion given in Section 1 of this unit, but now we are working inside $\mathbb{P}(\mathbf{R}^3)$ rather than $\mathbf{R}^3$ and we are also in a position to be less tentative about a few issues (for example, whether the pairs of lines have points in common).

In order to show that the line pairs have points in common, by the properties of $\mathbb{P}(\mathbf{R}^3)$ it is only necessary to check that they are coplanar. We shall only do this explicitly for the pair $\{\mathbf{AB}, \mathbf{A'B'}\}$, for the other cases follow by interchanging the roles of **A, B, C** and **A′, B′, C′.** Since $\mathbf{A'} \in \mathbf{QA}$ and $\mathbf{B'} \in \mathbf{QB}$, it follows that the lines **AB** and **A′B′** both line in the plane determined by **Q, A** and **B.** As noted in the second sentence of the paragraph, it follows similarly that $\{\mathbf{AC}, \mathbf{A'C'}\}$ and $\{\mathbf{BC}, \mathbf{B'C'}\}$ are coplanar pairs of lines.

By construction, we know that **D, E** and **F** — which are points on the lines **AB, AC,** and **BC** — lie in the plane **ABC** determined by those three points, and likewise they lie in the plane **A′B′C′.** If these two planes are distinct, then we know that **D, E** and **F** all lie on the line in which these planes intersect and we are done. To see that the planes are distinct, we shall assume they are the same and derive a contradiction. If they were distinct, then **A′** would lie on the plane of **ABC,** and thus the entire line $\mathbf{AA'} = \mathbf{QA}$ would lie on this plane; since **Q, A, B,** and **C** are not coplanar we know this cannot happen, and therefore **A′** does not lie on the plane determined by **ABC;** but this means that the planes **ABC** and **A′B′C′** cannot be the same.■

Euclidean interpretation. The nonplanar case of Desargues' Theorem immediately yields the following result in Euclidean geometry. Notice that the hypothesis is the same, but the conclusion involves three separate cases.

Theorem 2. *Suppose that we are given four noncoplanar points* $\mathbf{Q}, \mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbf{R}^3$, *and suppose we are given three other points* **A′, B′** *and* **C′** *such that* $\mathbf{A'} \in \mathbf{QA}$, $\mathbf{B'} \in \mathbf{QB}$, *and* $\mathbf{C'} \in \mathbf{QC}$. *Then* exactly *one of the following is true*:

(1) *The pairs of corresponding lines* $\{\mathbf{AB}, \mathbf{A'B'}\}$, $\{\mathbf{AC}, \mathbf{A'C'}\}$, *and* $\{\mathbf{BC}, \mathbf{B'C'}\}$ *meet at points* **D, E** *and* **F** *respectively, and these three points are collinear.*

(2) *Exactly one of the pairs of lines consists of parallel lines. Furthermore, in this case if* $\mathbf{AB} \parallel \mathbf{A'B'}$, *then these two lines are parallel to the line* **EF,** *where* **E** *is the common point of* **AC** *and* **A′C′,** *and* **F** *is the common point of* **BC** *and* **B′C′.**

(3) *All three of the pairs of lines are pairs of parallel lines; in other words, we have* $\mathbf{AB} \parallel \mathbf{A'B'}$, $\mathbf{AC} \parallel \mathbf{A'C'}$, *and* $\mathbf{BC} \parallel \mathbf{B'C'}$.

In the second case, note that one has analogous conclusions if $\mathbf{AC} \parallel \mathbf{A'C'}$ or $\mathbf{BC} \parallel \mathbf{B'C'}$, and these can be extracted by suitably interchanging the roles of the variables **A, B, C** and their counterparts **A′, B′, C′** in the proof.

The Euclidean version of Desargues' Theorem shows how projective geometry can provide an effective means for giving unified formulations and proofs of otherwise complicated statements in Euclidean geometry (the "bewildering chaos of special cases" in the Dieudonné quotation at the beginning of Section 2). The Euclidean interpretations of Desargues' Theorem when **Q** is the point at infinity are described in

Theorems $\mathbf{9}$ and $\mathbf{10}$ on page 129 of Ryan (strictly speaking, Ryan treats the planar rather than the nonplanar case, but the conclusion is the same in both cases).

<u>Comment on the proof.</u> All we need to do is to interpret the conclusions of the projective theorem in terms of Euclidean geometry. The conclusion of Desargues' Theorem applies to the extended ordinary lines, and the intersection points of these extended ordinary lines may be ordinary points or ideal points. The first case corresponds to the case in which they are all ordinary points, and the second to the case in which exactly one is an ideal point. Finally, if at least two of the intersection points are ideal points, then by the collinearity statement the third must also be an ideal point, so that the lines are parallel in pairs.■

Clearly one would expect that a proof of Desargues' Theorem without using $\mathbb{P}(\mathbf{R}^3)$ would involve three separate arguments for the individual cases.

Convenient choices for homogeneous coordinates

We are now going to discuss proofs of theorems in projective geometry using homogeneous coordinates. Frequently it is very helpful to choose the latter so that the algebraic computations become as simple as possible, and the next few results provide frequently used ways of doing so.

Frequently it is useful to have some sort of coding for passing back and forth between points in projective spaces and sets of homogeneous coordinates representing them. One method of doing so is to denote geometric points by ordinary capital letters and homogeneous coordinates by corresponding lower case Greek letters; since there are more letters in the Latin alphabet than the Greek alphabet, some additional adjustments are necessary; in the table we use two forms of the Greek letter <u>phi</u> and we insert Cyrillic characters <u>zhe</u>, <u>oborotnoye</u>, and <u>cha</u> for **J**, **U**, and **V**.

A	B	C	D	E	F	G	H	I	J	K	L	M
α	β	γ	δ	ε	ϕ	χ	θ	ι	ж	κ	λ	μ

N	O	P	Q	R	S	T	U	V	W	X	Y	Z
ν	o	φ	ψ	ρ	σ	τ	э	ч	ω	ξ	η	ζ

We then have the following three results on choices for homogeneous coordinates.

<u>Proposition 3.</u> *Let $n = 2$ or 3, let* **A** *and* **B** *be distinct points in $\mathbb{P}(\mathbf{R}^n)$, and let* **X** *be a third point which lies on the line* **AB**. *Then it is possible to choose homogeneous coordinates α, β, and ξ for the points* **A, B,** *and* **X** *such that we have $\xi = \alpha + \beta$. Furthermore, if α^*, β^*, and ξ^* are arbitrary homogeneous coordinates for such that $\xi^* = \alpha^* + \beta^*$, then there is a nonzero scalar k such that $\xi^* = k\xi$, $\alpha^* = k\alpha$, and $\beta^* = k\beta$.*

<u>Proposition 4.</u> *Let $n = 2$ or 3, let* **A, B,** *and* **C** *be noncollinear points in $\mathbb{P}(\mathbf{R}^n)$, and let* **X** *be a point in the plane* **ABC** *such that no three of the points* **A, B, C,** *and* **X** *are collinear. Then it is possible to choose homogeneous coordinates α, β, γ, and ξ for the points* **A, B, C,** *and* **X** *such that we have $\xi = \alpha + \beta + \gamma$. Furthermore, if α^*, β^*, γ^*, and*

ξ^* are arbitrary homogeneous coordinates for **A, B, C,** *and* **X** *for which we have $\xi^* = \alpha^* + \beta^* + \gamma^*$, then there is a nonzero scalar k such that $\xi^* = k\xi$, $\alpha^* = k\alpha$, $\beta^* = k\beta$, and $\gamma^* = k\gamma$.*

Proposition 5. *Let* **A, B, C,** *and* **D** *be noncoplanar points in $\mathbb{P}(\mathbf{R}^3)$, and let* **X** *be a point which such that no four of* **A, B, C, D,** *and* **X** *are coplanar. Then it is possible to choose homogeneous coordinates α, β, γ, δ, and ξ for the points* **A, B, C, D,** *and* **X** *such that we have $\xi = \alpha + \beta + \gamma + \delta$. Furthermore, if α^*, β^*, γ^*, δ^*, and ξ^* are arbitrary homogeneous coordinates for* **A, B, C, D,** *and* **X** *such that $\xi^* = \alpha^* + \beta^* + \gamma^* + \delta^*$, then there is a nonzero scalar k such that $\xi^* = k\xi$, $\alpha^* = k\alpha$, $\beta^* = k\beta$, $\gamma^* = k\gamma$, and $\delta^* = k\delta$.*

With a sufficiently abstract formal setting, all three of these would be cases of a single result; we have chosen to set things up in a more elementary manner to limit the time spent on projective geometry and to keep everything from becoming too abstract and heavily loaded with definitions.

Proofs. We shall prove these three theorems in the order they are stated above.

Suppose that **A** and **B** are distinct points in $\mathbb{P}(\mathbf{R}^n)$, and let **X** be a third point which lies on the line **AB**. Let α' and β' be arbitrary homogeneous coordinates for **A** and **B** respectively. Since **X** lies on **AB**, we know that a set ξ of homogeneous coordinates for **X** must be a linear combination of α' and β', so write $\xi = z\alpha' + u\beta'$ for suitable scalars z and u. We claim that both of these coefficients are nonzero; certainly both cannot be equal to zero, and if one is equal to zero then we have **X** = **A** or **X** = **B**. Therefore we know that $\alpha = z\alpha'$ and $\beta = u\beta'$ also represent **A** and **B** respectively, and for these choices of homogeneous coordinates we clearly have $\xi = \alpha + \beta$.

To prove the uniqueness statement for the first result, note that if ξ^* is any other set of homogeneous coordinates for **X**, then $\xi^* = k\xi$ for some nonzero scalar k and hence we have $\xi^* = k\xi = k\alpha + k\beta$. Since we also know there are nonzero scalars p and q such that $\alpha^* = p\alpha$ and $\beta^* = q\beta$, it follows that

$$k\alpha + k\beta = k\xi = \xi^* = \alpha^* + \beta^* = p\alpha + q\beta.$$

The assumption that **A** and **B** are distinct implies that α and β are linearly independent, and the latter in turn implies that the coefficients of α and β on the left and right hand expressions in the display above must be equal. Therefore $k = p = q$, so that $\alpha^* = k\alpha$ and $\beta^* = k\beta$.

Next, suppose that **A, B,** and **C** are noncollinear points in $\mathbb{P}(\mathbf{R}^n)$, and let **X** be a point in the plane **ABC** such that no three of the points **A, B, C,** and **X** are collinear. Let α', β', and γ' be arbitrary homogeneous coordinates for **A, B,** and **C** respectively. We then know that α', β', and γ' are linearly independent and hence that a set ξ of homogeneous coordinates for **X** must be a linear combination of α', β', and γ'; therefore we may write $\xi = z\alpha' + u\beta' + v\gamma'$ for suitable scalars z, u, and v. As in the preceding paragraph, we claim that all of these coefficients are nonzero. If, say, we had $z = 0$ then **X** would lie on the line **BC,** and hence z is nonzero; similar considerations show that the other two coefficients are nonzero. Therefore we know that $\alpha = z\alpha'$, $\beta = u\beta'$, and $\gamma = v\gamma'$ also represent **A, B,** and **C** respectively, and for these choices of homogeneous coordinates we clearly have $\xi = \alpha + \beta + \gamma$.

To prove the uniqueness statement for the second result, note again that if ξ^* is any other set of homogeneous coordinates for **X,** then $\xi^* = k\xi$ for some nonzero scalar k and therefore $\xi^* = k\xi = k\alpha + k\beta + k\gamma$. Since we also know there are nonzero scalars p, q and x such that $\alpha^* = p\alpha$, $\beta^* = q\beta$, and $\gamma^* = x\gamma$, it follows that

$$k\alpha + k\beta + k\gamma = k\xi = \xi^* = \alpha^* + \beta^* + \gamma^* = p\alpha + q\beta + x\gamma.$$

The assumption that **A, B,** and **C** are noncollinear implies that α, β, and γ are linearly independent, and the latter in turn implies that the coefficients of α, β, and γ on the left and right hand sides of the equation above must be equal. Therefore $k = p = q = x$, so that $\alpha^* = k\alpha$, $\beta^* = k\beta$, and $\gamma^* = k\gamma$.

Finally, suppose that **A, B, C,** and **D** be noncoplanar points in $\mathbb{P}(\mathbf{R}^3)$, and let **X** be a point which such that no four of **A, B, C, D,** and **X** are coplanar. Let α', β', γ', and δ' be arbitrary homogeneous coordinates for **A, B, C,** and **D** respectively. In this case we know that a set ξ of homogeneous coordinates for **X** must be a linear combination of α', β', γ', and δ', so we have $\xi = z\alpha' + u\beta' + v\gamma' + w\delta'$ for suitable scalars z, u, v, and w. As before, if any of these scalars were nonzero, then **X** would lie in a plane three of the other points, so all four coefficients must be nonzero and therefore $\alpha = z\alpha'$, $\beta = u\beta'$, $\gamma = v\gamma'$, and $\delta = w\delta'$ also represent **A, B, C,** and **D** respectively, and for these choices of homogeneous coordinates we have $\xi = \alpha + \beta + \gamma + \delta$.

To prove the uniqueness statement for the last result, note again that if ξ^* is any other set of homogeneous coordinates for **X,** then $\xi^* = k\xi$ for some nonzero scalar k and therefore $\xi^* = k\xi = k\alpha + k\beta + k\gamma + k\delta$. Since we also know there are nonzero scalars p, q and x such that $\alpha^* = p\alpha$, $\beta^* = q\beta$, $\gamma^* = x\gamma$, and $\delta^* = y\delta$, it follows that

$$k\alpha + k\beta + k\gamma + k\delta = k\xi = \xi^* = \alpha^* + \beta^* + \gamma^* + \delta^* = p\alpha + q\beta + x\gamma + y\delta.$$

The assumption that **A, B, C,** and **D** are noncollinear implies that α, β, γ, and δ are linearly independent, and the latter in turn implies that the coefficients of α, β, γ, and δ on the left and right hand sides of the equation above must be equal. Therefore we have $k = p = q = x = y$ as well as the corresponding vector equations $\alpha^* = k\alpha$, $\beta^* = k\beta$, $\gamma^* = k\gamma$, and $\delta^* = k\delta$.■

The following proof for the planar case of Desargues' Theorem illustrates how good choices for homogeneous coordinate can simplify the details in some computational arguments.

Theorem 6. (Desargues' Theorem — planar case). *Suppose that we are given four* ***coplanar*** *points* **Q, A, B, C** $\in$ $\mathbb{P}(\mathbb{R}^3)$ *such that* **A, B,** *and* **C** *are noncollinear, and suppose we are given three other points* **A′, B′** *and* **C′** *such that* **A′** $\in$ **QA, B′** $\in$ **QB,** *and* **C′** $\in$ **QC.** *Then the pairs of corresponding lines* **{AB, A′B′}, {AC, A′C′},** *and* **{BC, B′C′}** *meet at points* **D, E** *and* **F** *respectively, and these three points are collinear.*

(**NOTE:** The previous drawing for the noncoplanar case also applies equally well to the coplanar case; no drawing is included here because there are already three figures for Desargues' Theorem in these notes.)

Proof. Let ψ, α, β, and γ be homogeneous coordinates for **Q, A, B,** and **C** respectively. Since **A′** is a third point on the line **QA** we know that homogeneous coordinates **A′** are given by $p\,\psi + q\,\alpha$, and as in the proof of the preceding theorem we know that both p and q must be nonzero. If we multiply these homogeneous coordinates by p^{-1}, we obtain a new set of homogeneous coordinates ξ for **A′** of the form $\psi + x\,\alpha$, where x is nonzero. Similarly, one can find homogeneous coordinates η and ζ for **B′** and **C′** of the forms $\psi + y\,\beta$ and $\psi + z\gamma$, where y and z are nonzero.

Since $(\psi + y\,\beta) - (\psi + z\gamma) = y\,\beta - z\gamma$, it follows that this vector gives homogeneous coordinates ϕ for the point **F** where **BC** and **B′C′** meet; similar considerations show that the intersection points **D** and **E** have homogeneous coordinates δ and ε which are given by the vectors $x\,\alpha - y\,\beta$ and $x\,\alpha - z\gamma$ respectively. Since $\varepsilon = \delta + \phi$, it follows that the points **D, E** and **F** must be collinear.■

There is also a corresponding version of Desargues' Theorem for coplanar points in Euclidean geometry:

Theorem 7. *Suppose that we are given four* ***coplanar*** *points* **Q, A, B, C** $\in$ $\mathbb{R}^3$ *such that* **A, B,** *and* **C** *are noncollinear, and suppose we are given three other points* **A′, B′** *and* **C′** *such that* **A′** $\in$ **QA, B′** $\in$ **QB,** *and* **C′** $\in$ **QC.** *Then* exactly *one of the following is true*:

(1) *The pairs of corresponding lines* **{AB, A′B′}, {AC, A′C′},** *and* **{BC, B′C′}** *meet at points* **D, E** *and* **F** *respectively, and these three points are collinear.*

(2) *Exactly one of the pairs of lines consists of parallel lines. Furthermore, in this case if* **AB || A′B′,** *then these two lines are parallel to the line* **EF,** *where* **E** *is the common point of* **AC** *and* **A′C′,** *and* **F** *is the common point of* **BC** *and* **B′C′.**

(3) *All three of the pairs of lines are pairs of parallel lines; in other words, we have* **AB || A′B′, AC || A′C′,** *and* **BC || B′C′.**

This can be derived from the coplanar projective version of Desargues' Theorem in the same way that its noncoplanar analog was derived from the noncoplanar projective version of Desargues' Theorem.■

The $2-$dimensional Euclidean interpretations of Desargues' Theorem when **Q** is a point at infinity are described in Theorems 9 and 10 on page 129 of Ryan.

Having devoted a section of these notes to duality, it is hard to avoid the following:

<u>Question.</u> *What happens if we* ***dualize*** *Desargues' Theorem in* $\mathbb{P}(\mathbb{R}^2)$**?**

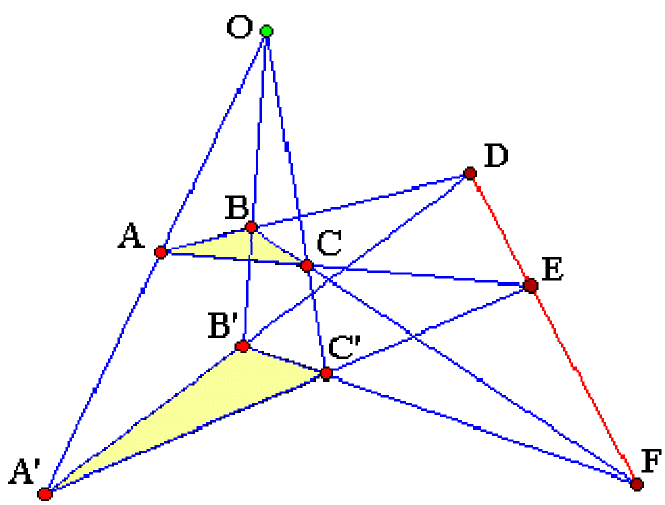
The duals of the two triples of noncollinear points **A, B, C** and **A′, B′, C′** will be two triples of nonconcurrent lines that we shall denote by **L, M, N** and **L′, M′, N′.** The condition that the lines **AA′, BB′,** and **CC′** all pass through a point **Q** dualizes to a condition that the points determined by the three line intersections **L ∩ L′, M ∩ M′,** and **N ∩ N′** are collinear. Let us call these points **T, U,** and **V** respectively. The conclusion that three associated points be collinear dualizes to a statement that three associated lines be concurrent. More precisely if we take

$$\mathbf{X} \in \mathbf{M} \cap \mathbf{N}, \quad \mathbf{Y} \in \mathbf{L} \cap \mathbf{N}, \quad \mathbf{Z} \in \mathbf{L} \cap \mathbf{M}$$

$$\mathbf{X'} \in \mathbf{M'} \cap \mathbf{N'}, \quad \mathbf{Y'} \in \mathbf{L'} \cap \mathbf{N'}, \quad \mathbf{Z} \in \mathbf{L'} \cap \mathbf{M'}$$

then the conclusion of Desargues' Theorem dualizes to an assertion that the lines **XX′, YY′,** and **ZZ′** are concurrent.

What happens if we draw a figure to illustrate the dual conditions? It turns out that we get exactly the same configuration as in Desargues' Theorem with all the points renamed. For the sake of convenience, we reproduce a figure from Section 1 below. In the dual setting described above, the points **T, U,** and **V** correspond to the points **D, E,** and **F** which turn out to be collinear, and the six points **X, Y, Z, X′, Y′, Z′** respectively correspond to the points **A, B, C, A′, B′, C′.**



The conclusion of the dualized theorem states that the lines **X X′, YY′,** and **Z Z′** are concurrent. Of course, this looks very much like an assumption in Desargues' Theorem. To obtain more insight into the relationship between the theorem and its dual, consider the following reformulation of Desargues' Theorem in the projective plane:

> *Suppose we are given two triples of noncollinear points* **A, B, C** *and* **A′, B′, C′** *such that all six points are distinct. If the lines* **AA′, BB′,** *and* **CC′**

are concurrent, then the three points in the line intersections **AB** $\cap$ **A′B′, AC** $\cap$ **A′C′**, *and* **BC** $\cap$ **B′C′** *are collinear.*

The dual of Desargues' Theorem then has the following corresponding formulation.

<u>**Theorem 8.** (Planar dual of Desargues' Theorem)</u> *Suppose we are given two triples of noncollinear points* **A, B, C** *and* **A′, B′ C′** *such that all six points are distinct. If the points in the line intersections* **AB** $\cap$ **A′B′, AC** $\cap$ **A′C′**, *and* **BC** $\cap$ **B′C′** *are collinear, then the three lines* **AA′, BB′**, *and* **CC′** *are concurrent.*■

In other words, we have shown that ***the dual of Desargues' Theorem in the plane is essentially its converse,*** so that *this converse is also true in* $\mathbb{P}(\mathbf{R}^2)$ *by duality.*■

Desargues' Theorem plays an extremely fundamental role in projective geometry, but an explanation of this fact would go far beyond the scope of this course. Further information appears in the previously cited directory

http://math.ucr.edu/~res/progeom

and many standard references for projective geometry.

The (Hexagon) Theorem of Pappus

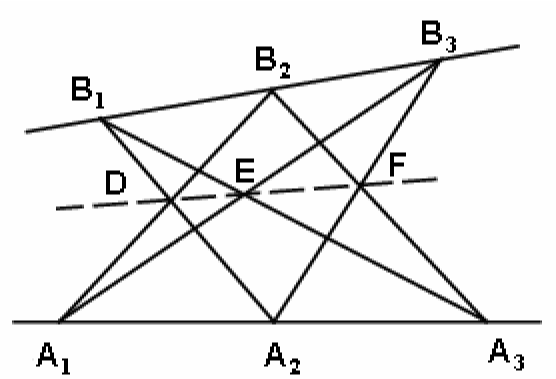
We have already noted that that the statement of Desargues' Theorem does not involve measurements, and in fact its proof in the noncoplanar case also does not use anything about measurements (one can also give a measurement – free proof for planes inside projective **3** – space but this requires more work; one reference is Wallace and West, ***Roads to Geometry,*** 3rd Ed., pp. 354 – 360). Projective geometry deals mainly with such results involving the positioning and placement of geometrical figures. Probably the earliest result of this sort in geometry was discovered by Pappus of Alexandria in the 4th century A. D., and it also plays an important role in projective geometry (again for reasons outside the scope of this course). Frequently this result is called ***Pappus' Theorem***, but since this name is also used for certain other results (for example, the theorems studied in first year calculus which involve the centroids for solids and surfaces of revolution), it is sometimes useful to add the term "Hexagon" in order to avoid ambiguities.

<u>**Theorem 9.** (Pappus' Hexagon Theorem)</u> *Suppose* $\{A_1, A_2, A_3\}$ *and* $\{B_1, B_2, B_3\}$ *are triples of noncollinear points in* $\mathbb{P}(\mathbf{R}^2)$; *assume that the two lines and six points are distinct. Then the cross intersection points*

$$D \in A_2B_3 \cap A_3B_2, \qquad E \in A_1B_3 \cap A_3B_1, \qquad F \in A_1B_2 \cap A_2B_1$$

are collinear.

A drawing to illustrate Pappus' Hexagon Theorem appears on the next page.



(Source: http://mathworld.wolfram.com/PappussHexagonTheorem.html)

As before, if we take the original six points to be ordinary points in the Euclidean plane, then the conclusion breaks down into three separate cases. It is possible to prove the Euclidean result by classical methods, but once again it is by no means easy to do so.

Proof. At most one of the six points lies on both lines. If we permute the indexing variables $\{1, 2, 3\}$ we can arrange things so that any common point would be either $\mathbf{A_3}$ or $\mathbf{B_3}$, and hence we might as well assume that none of the points $\mathbf{A_1}$, $\mathbf{A_2}$, $\mathbf{B_1}$, $\mathbf{B_2}$ lie on both lines. It follows that no three of these points are collinear.

By the coordinate choice theorem for four points, we may choose homogeneous coordinates $\alpha_1, \alpha_2, \beta_1, \beta_2$ for $\mathbf{A_1}, \mathbf{A_2}, \mathbf{B_1}, \mathbf{B_2}$ such that $\beta_2 = \alpha_1 + \alpha_2 + \beta_1$. Since $\mathbf{A_3}$ lies on the line $\mathbf{A_1A_2}$ and $\mathbf{B_3}$ lies on the line $\mathbf{B_1B_2}$, as in the proof of the planar Desargues' Theorem we can find homogeneous coordinates α_3 and β_3 for $\mathbf{A_3}$ and $\mathbf{B_3}$ such that

$$\alpha_3 = \alpha_1 + p\,\alpha_2, \qquad \beta_3 = \beta_1 + q\,\beta_2$$

for suitable scalars p and q. Also, since $\mathbf{F} \in \mathbf{A_1B_2} \cap \mathbf{A_2B_1}$, we know there are scalars x, y, u, v such that homogeneous coordinates ϕ for $\mathbf{F}$ are given by

$$\phi' = u\,\beta_1 + v\,\alpha_2 = x\,\alpha_1 + y\,\beta_2 = (x+y)\,\alpha_1 + y\,\alpha_2 + y\,\beta_1 .$$

Equating the coefficients of the expressions on the left and right hand sides, we obtain the relations $x + y = 0$ and $y = u = v$. Therefore $\mathbf{F}$ has homogeneous coordinates ϕ given by $\beta_1 + \alpha_2$. Similarly, since $\mathbf{E} \in \mathbf{A_1B_3} \cap \mathbf{A_3B_1}$, we may write homogeneous coordinates for $\mathbf{E}$ in the form

$$\varepsilon' = x\,\alpha_1 + y\,\beta_3 = u\,\alpha_3 + v\,\beta_1$$

for suitable scalars, and if we substitute the previously specified values for β_3 and α_3 we obtain the following equations:

$$x\,\alpha_1 + y\,\beta_1 + q\,y\,\beta_2 = (x + q\,y)\,\alpha_1 + q\,y\,\alpha_2 + (y + q\,y)\,\beta_1 = u\,\alpha_1 + p\,u\,\alpha_2 + v\,\beta_1$$

Equating coefficients as before, we find that $\mathbf{E}$ has homogeneous coordinates $\boldsymbol{\varepsilon}$ given by $\alpha_1 + p\,\alpha_2 + (1 + q^{-1}p)\,\beta_1$. Yet another calculation of the same type shows that $\mathbf{D}$ has homogeneous coordinates $\boldsymbol{\delta}$ given by $\alpha_1 + (1 + q^{-1} - q^{-1}p)\,\alpha_2 + (1 + q^{-1})\,\beta_1$. It then follows that

$$\delta - \varepsilon = (p - 1 - q^{-1} + pq - 1)\,\alpha_2 + (p + q^{-1}p - 1 - q^{-1})\,\beta_1$$

is a multiple of $\phi = \beta_1 + \alpha_2$. Since ϕ is a set of homogeneous coordinates for **F,** it follows that the three points **D, E** and **F** are collinear.■

By duality, the preceding argument also yields the corresponding dual statement.

__**Theorem 10.**__ __(Dual of Pappus' Hexagon Theorem)__ *Let* $\{\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3\}$ *and* $\{\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3\}$ *be triples of nonconcurrent lines in* $\mathbb{P}(\mathbf{R}^3)$; *assume that the two points and six lines are distinct. Let* $\mathbf{C}_{i,j}$ *be the common point of* $\mathbf{L}_i$ *and* $\mathbf{M}_j$. *Then the lines*

$$\mathbf{A}_{2,3}\mathbf{A}_{3,2}, \qquad \mathbf{A}_{1,3}\mathbf{A}_{3,1}, \qquad \mathbf{A}_{1,2}\mathbf{A}_{2,1}$$

are concurrent.■

Appendix – Pascal's Theorem

In several other languages Pappus' Hexagon Theorem is often called ***Pascal's Theorem*** because it may be viewed as a singular case of a result discovered by B. Pascal (1623 – 1662). In order to state the result we need to extend the notion of a conic section to the projective plane.

__**Definition.**__ A subset Γ of $\mathbb{P}(\mathbf{R}^2)$ is said to be a ***conic*** (or ***conic section***) if is the set of points whose homogeneous coordinates **x** satisfy a second degree equation of the form ${}^{\mathbf{T}}\mathbf{x}\mathbf{A}\mathbf{x} = 0$ for some symmetric 3×3 matrix **A.**

Before proceeding, we shall dispose of two elementary issues.

__**Proposition 11.**__ **(1)** *If one set* ξ *of homogeneous coordinates for a point* **X** *satisfies an equation of the form* ${}^{\mathbf{T}}\mathbf{x}\mathbf{A}\mathbf{x} = 0$ (*where* **A** *is not necessarily symmetric*), *then all sets of homogeneous coordinates for* **X** *also satisfy this equation.*

(2) *If* Γ *is the set of points whose homogeneous coordinates satisfy an equation of the form* ${}^{\mathbf{T}}\mathbf{x}\mathbf{A}\mathbf{x} = 0$ *where* **A** *is not necessarily symmetric, then there is a symmetric matrix* **B** *such that* Γ *is also the set of points whose homogeneous coordinates satisfy the equation* ${}^{\mathbf{T}}\mathbf{x}\mathbf{B}\mathbf{x} = 0$.

__***Proof.***__ We begin with the first part. If ξ is a set of homogeneous coordinates for X then every other set is given by $k\xi$ where k is a nonzero scalar. Since ${}^{\mathbf{T}}\xi\mathbf{A}\xi = 0$, we have

$${}^{\mathbf{T}}(k\xi)\,\mathbf{A}\,(k\xi) \;=\; k^2\left[{}^{\mathbf{T}}(k\xi)\,\mathbf{A}\,(k\xi)\right] \;=\; k^2\,0 \;=\; 0$$

and hence the equation is satisfied by an arbitrary set of homogeneous coordinates for the point **X.**

Suppose Γ is the set of all **x** satisfying ${}^{\mathbf{T}}\mathbf{x}\mathbf{A}\mathbf{x} = 0$ where **A** is not necessarily symmetric, and consider the following transposition identity:

$${}^{\mathbf{T}}\left[{}^{\mathbf{T}}\mathbf{x}\mathbf{A}\mathbf{x}\right] \;=\; {}^{\mathbf{T}}\mathbf{x}\,{}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}({}^{\mathbf{T}}\mathbf{x}) \;=\; {}^{\mathbf{T}}\mathbf{x}({}^{\mathbf{T}}\mathbf{A})\mathbf{x}$$

Since the objects in these equations are all $\mathbf{1 \times 1}$ matrices and every such matrix is equal to its transpose, it follows that ${}^{\mathbf{T}}\mathbf{x}\mathbf{A}\mathbf{x} = {}^{\mathbf{T}}\mathbf{x}({}^{\mathbf{T}}\mathbf{A})\mathbf{x}$, which means that one of these is zero if and only if the other is zero. Set **B** equal to the symmetric matrix $\mathbf{A} + {}^{\mathbf{T}}\mathbf{A}$; by the previous discussion we have

$$ {}^{\mathbf{T}}\mathbf{x}\mathbf{B}\mathbf{x} \;=\; {}^{\mathbf{T}}\mathbf{x}\mathbf{A}\mathbf{x} + {}^{\mathbf{T}}\mathbf{x}({}^{\mathbf{T}}\mathbf{A})\mathbf{x} \;=\; 2\left[{}^{\mathbf{T}}\mathbf{x}\mathbf{A}\mathbf{x}\right] $$

so that ${}^{\mathbf{T}}\mathbf{x}\mathbf{A}\mathbf{x} = \mathbf{0}$ if and only if ${}^{\mathbf{T}}\mathbf{x}\mathbf{B}\mathbf{x} = \mathbf{0}$.■

We should also note that ***every ordinary conic section in $\mathbf{R}^2$ determines a projective conic in $\mathbb{P}(\mathbf{R}^2)$;*** the latter is often described as a ***projectivization*** of the original conic. To illustrate the assertion about projective versions of ordinary conics, if we are given a conic in $\mathbf{R}^2$ defined by a quadratic equation in two variables

$$ Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F \;=\; 0 $$

then the set of points on this conic is the set of ordinary points whose homogeneous coordinates satisfy the homogeneous quadratic equation

$$ Ax_1^{\,2} + 2Bx_1\,x_2 + Cx_2^{\,2} + 2Dx_1x_3 + 2Ex_2\,x_3 + Fx_3^{\,2} \;=\; 0 $$

and the latter is just the set of solutions for the equation ${}^{\mathbf{T}}\mathbf{x}\mathbf{Q}\mathbf{x} = \mathbf{0}$, where **Q** is the following symmetric $\mathbf{3 \times 3}$ matrix:

$$ \begin{pmatrix} A & B & D \\ B & C & E \\ D & E & F \end{pmatrix} $$

Definition. A conic is said to be ***nonsingular*** if one can choose the symmetric matrix to be invertible.

Examples. The standard nontrivial conic sections in the plane determine nonsingular projective conics in the sense of the definition above. For example, the standard unit circle $x^2 + y^2 - 1 = 0$, the hyperbola $x^2 - y^2 - 1 = 0$, and the parabola $x^2 - 4y = 0$ determine the projective conics defined by following invertible matrices:

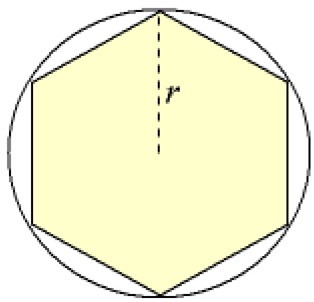
$$ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{pmatrix} $$

We have already noted that the projectivization of an ordinary conic consists of the conic itself and possibly some ideal points, and thus it is natural to ask how many points are added when one passes to the projectivizations of the examples described above. It is easy to work this out using the numerical information given above (and the fact that ideal points are those whose third homogeneous coordinates are zero), and in fact the circle has no ideal points while the hyperbola has two and the parabola has one.

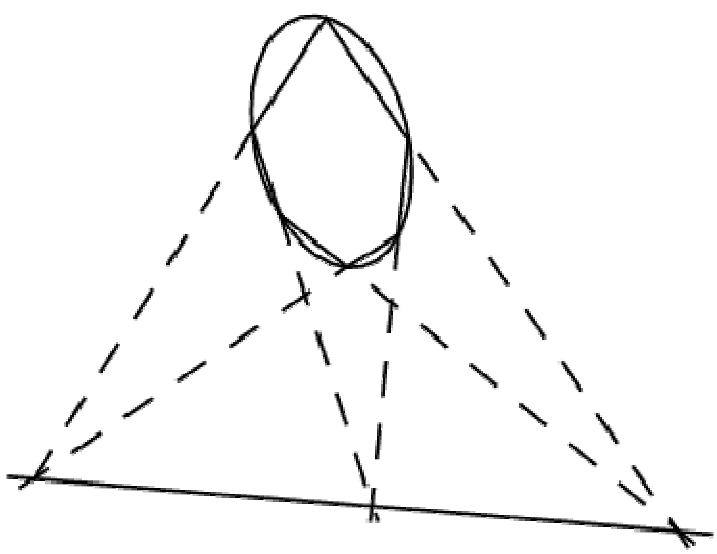
Conics in projective geometry. Everyday experience shows that the photographic image of a circle or ellipse is normally an ellipse (in some exceptional cases the image is a circle, and in still others it may be a line); therefore, it is not surprising that conics are objects of interest in projective geometry. In fact, a large amount of work has been done on conics in the projective plane and their generalizations (for example, to projective quadrics in projective $3-$space) and such conics and quadrics have many important properties, but we shall limit ourselves to stating the theorem of Pascal that was mentioned above.

<u>**Theorem 12.** (Pascal's hexagon theorem for conics)</u> *Let* Γ *be a nonsingular conic in* $\mathbb{P}(\mathbb{R}^2)$, *and let* $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4, \mathbf{A}_5, \mathbf{A}_6$ *be points on* Γ. *Then the intersection points* $\mathbf{A}_1\mathbf{A}_2 \cap \mathbf{A}_4\mathbf{A}_5$, $\mathbf{A}_2\mathbf{A}_3 \cap \mathbf{A}_5\mathbf{A}_6$, $\mathbf{A}_3\mathbf{A}_4 \cap \mathbf{A}_6\mathbf{A}_1$ *are collinear.*

<u>Examples.</u> If Γ is a circle and we are given an inscribed regular hexagon whose vertices are $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4, \mathbf{A}_5, \mathbf{A}_6$ (in that order), then $\mathbf{A}_1\mathbf{A}_2 \parallel \mathbf{A}_4\mathbf{A}_5$, $\mathbf{A}_2\mathbf{A}_3 \parallel \mathbf{A}_5\mathbf{A}_6$, and $\mathbf{A}_3\mathbf{A}_4 \parallel \mathbf{A}_6\mathbf{A}_1$; in this case the intersection points of the extended lines are all ideal points, and the validity of Pascal's Theorem in this case can be checked directly because the three intersection points all lie on the line at infinity.

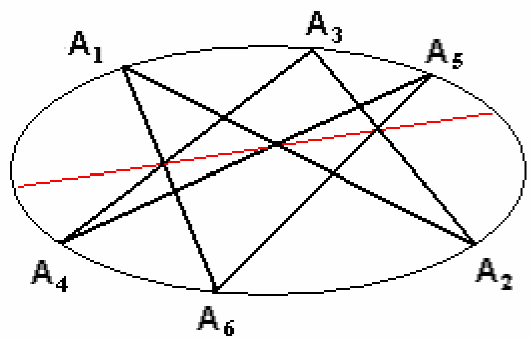


A drawing to illustrate a more typical case of Pascal's Theorem is given below.



(Source: http://mathworld.wolfram.com/PascalsTheorem.html)

Pappus' Theorem is related to Pascal's Theorem because a pair of intersecting lines can be viewed as a singular conic (for example, the solutions of the ordinary quadratic equation $x^2 - y^2 = 0$ are the points on the lines $y = \pm x$). If we view $\{\mathbf{A}_1, \mathbf{A}_3, \mathbf{A}_5\}$ as the triple of points on one of these lines and $\{\mathbf{A}_4, \mathbf{A}_6, \mathbf{A}_2\}$ as the triple of points on the other, then one can view Pascal's Theorem as an analog of Pappus' Theorem for nonsingular conics. The drawing below illustrates the analogy very clearly.



This drawing also illustrates an important feature of Pascal's Theorem. Namely, there is no requirement that the hexagon $\mathbf{A_1A_2A_3A_4\,A_5A_6}$ be a convex polygon, and there is even no requirement that the sides of this "generalized hexagon" must meet only at common vertices. In projective geometry, a "hexagon" normally refers to the union of the six relevant lines: $\mathbf{A_1A_2 \cup A_2A_3 \cup A_3A_4 \cup A_4A_5 \cup A_5A_6 \cup A_6A_1}$

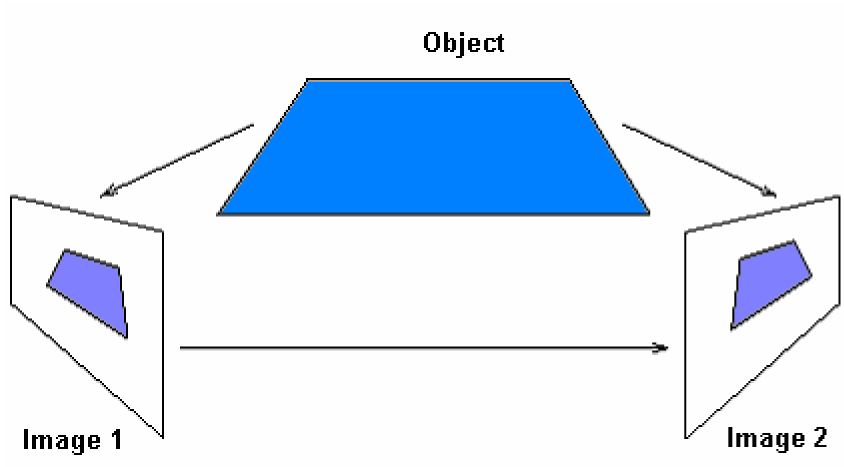
IV.6 : Cross ratios and projective collineations

In the final section of this unit we shall return to some issues involving perspective projections, and we shall also discuss projective analogs of the affine transformations on $\mathbf{R}^n$ that were introduced in Section **II.4.** Important relationships between such transformations and perspective projections will also be discussed.

Perspective invariance

One obvious question about the relation of a picture to its image is which features of the object are preserved by the picture and which are not. Several facts are obvious even if one does not think about the theory of perspective drawing in terms of mathematics. For example, it is clear that collinear points go to collinear points (and we have proved this mathematically in Section 1), but both absolute and relative distances are often badly distorted. In particular, it **C** is the midpoint of **A** and **B** in the original object, the image of **C** is generally not equal to the midpoint of the images of **A** and **B,** and in fact the image of the midpoint can be nearly anywhere on the segment joining the images of **A** and **B.** In the 15th century, the previously mentioned artist/writer Alberti raised a question that turns out to be important both practically and theoretically:

> *If two different projections of an object are given, what properties are the same in both images?*



(Source: http://homepages.inf.ed.ac.uk/rbf/CVonline/LOCAL_COPIES/BEARDSLEY/node3.html)

As before, we know that lines are preserved but that distances can be badly distorted. In particular, the midpoint of the images of two points for one projection need not be the midpoint of the two corresponding image points in the other.

Cross ratios

If we are given two points **a** and **b** in the Euclidean plane or $3-$ space and **x** is a point on the line **ab**, then we say that **x** *divides* **a** *and* **b** *in the ratio* $(1-t):t$ if we have the equation **x** $=$ **b** $+\, t($**a** $-$ **b**$)$ or equivalently **x** $= (1-t)$**b** $+\, t$**a**. Likewise, if p and q are any real numbers, then we say that **x** *divides* **a** *and* **b** *in the ratio* $p:q$ if (p,q) is a nonzero multiple of $(1-t, t)$, where t is given as before. Note that if **x** is between **a** and **b** then it follows that **x** divides **a** and **b** in the ratio $d(\mathbf{a},\mathbf{x}):d(\mathbf{b},\mathbf{x})$.

As noted before, if we are given a perspective projection Ψ and three collinear points **a**, **b** and **x** such that **x** divides **a** and **b** in the ratio $p:q$, then one cannot draw any general conclusions about the ratio in which $\Psi(\mathbf{x})$ divides $\Psi(\mathbf{a})$ and $\Psi(\mathbf{b})$. However, if **a, b, c** and **d** are four collinear points one can define a number called the ***cross ratio*** of these points which does ***not*** change under perspective transformation. As we shall see, the Euclidean definition is complicated, but it is straightforward to give a definition in terms of projective geometry.

Definition. Let $n = 2$ or 3, and let **A, B, C,** and **D** be distinct collinear points in $\mathbb{P}(\mathbf{R}^n)$. Choose homogeneous coordinates α, β, and γ for **A, B,** and **C** such that $\gamma = \alpha + \beta$, so that homogeneous coordinates for **D** are given by $u\,\alpha + v\,\beta$, where u and v are nonzero scalars. The ***cross ratio*** **(ABCD)** is defined to be the quotient u/v. Frequently some punctuation marks appear between consecutive points, with notation like **(A, B, C, D)** or **(A, B; C, D)**.

The preceding definition involves some choices for homogeneous coordinates, so before using it we must prove that ***the cross ratio defined above remains unchanged if we make different choices of homogeneous coordinates.***

First of all, we shall confirm that the cross ratio does not depend upon the choice of homogeneous coordinates at the last step; if δ' is another set of homogeneous coordinates for **D,** then we have $\delta' = q\delta$ for some nonzero scalar q, which means that $\delta' = qu\alpha + qv\beta$ and the corresponding ratio is qu/qv, which is equal to the previously computed ratio u/v. Next, we confirm that the value does not depend upon the initial choices for homogeneous coordinates for **A, B,** and **C.** By the results of the preceding section, if we make any other such choices α^*, β^*, γ^* then there is a nonzero scalar k such that $\alpha^* = k\alpha$, $\beta^* = k\beta$, *and* $\gamma^* = k\gamma$. Suppose now that β^* is an arbitrary set of homogeneous coordinates for **D.** Then we have $\delta^* = x\alpha^* + y\beta^*$ for suitable scalars x and y, but we also know that $\delta^* = x\alpha^* + y\beta^* = kx\alpha + ky\beta$, and since δ^* is a nonzero scalar multiple $z\delta$ of the previous homogeneous coordinates for **D,** it follows that $xk\alpha + yk\beta = zu\alpha + zv\beta$. Equating coefficients, we have $xk = zu$ and $yk = zv$, and therefore we also have $u/v = zu/zv = kx/ky = x/y$, showing that the value obtained with the new choices agrees with the value obtained with the old ones.■

The following realization property for cross ratios is simple but important for many purposes.

<u>**Proposition 1.**</u> *Let* $n = 2$ *or* 3, *let* **A, B,** *and* **C** *be distinct collinear points in* $\mathbb{P}(\mathbb{R}^n)$, *and let* k *be an arbitrary nonzero scalar. Then there is a unique point* **D** *on the line of* **A, B,** *and* **C** *such that* **D** *is distinct from the three original points and* **(ABCD)** $= k$.

<u>Proof.</u> It is easy to see that such a point exists. Choose homogeneous coordinates α, β, and γ for **A, B,** and **C** such that $\gamma = \alpha + \beta$, and take **D** to be the point represented by the homogeneous coordinates $\delta = k\alpha + \beta$. To prove uniqueness, let **E** be an arbitrary point such that **(ABCE)** $= k$, and let **D** be given as in the existence statement. Then homogeneous coordinates for **E** are given by $\varepsilon = x\alpha + y\beta$ for suitable nonzero scalars x and y, and the cross ratio condition implies that $k = x/y$, or equivalently $ky = x$. Therefore we have $\varepsilon = ky\alpha + y\beta = k\delta$, so that **d** and **e** represent the same point in $\mathbb{P}(\mathbb{R}^n)$ and hence **D** = **E.**■

There are 24 different orders in which four distinct collinear points **A, B, C, D** may be arranged, and one obvious problem is to determine what happens to the cross ratio if the points are rearranged. The answer is given by the following result:

<u>**Theorem 2.**</u> *Let* **A, B, C,** and **D** *be distinct collinear points, and assume that the cross ratio* **(ABCD)** *is equal to* k. *Then the cross ratios for the rearrangements of the four points are given as follows*:

$$\begin{aligned}
k &= (\mathbf{ABCD}) = (\mathbf{BADC}) = (\mathbf{CDAB}) = (\mathbf{DCBA})\\
1/k &= (\mathbf{ABDC}) = (\mathbf{BACD}) = (\mathbf{DCAB}) = (\mathbf{CDBA})\\
(1-k) &= (\mathbf{ACBD}) = (\mathbf{CADB}) = (\mathbf{BDAC}) = (\mathbf{DBCA})\\
1/(1-k) &= (\mathbf{ACDB}) = (\mathbf{CABD}) = (\mathbf{DBAC}) = (\mathbf{BDCA})\\
(1-k)/k &= (\mathbf{ADBC}) = (\mathbf{DACB}) = (\mathbf{BCAD}) = (\mathbf{CBDA})\\
k/(1-k) &= (\mathbf{ADCB}) = (\mathbf{DABC}) = (\mathbf{CBAD}) = (\mathbf{BCDA})
\end{aligned}$$

The proof of this result is a sequence of elementary and eventually boring computations, and it is left to the exercises; hints are given in the latter for minimizing the amount of computations needed to complete the proof.■

The next result gives a standard and often useful formula for the cross ratio.

Theorem 3. *Suppose that* $\mathbf{D_1}$, $\mathbf{D_2}$, $\mathbf{D_3}$, *and* $\mathbf{D_4}$ *are distinct points on the line containing the three points* **A, B,** *and* **C,** *and for each index value* m *suppose that* $(\mathbf{ABCD}_m) = z_m$. *Then the cross ratio* $(\mathbf{D_1\,D_2\,D_3\,D_4})$ *is given by the following expression*:

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

The name of the cross ratio suggests it should be somehow related to the ratio in which a point **x** on the line **ab** divides the two points **a** and **b,** and the next result shows that one can view the cross ratio **(abcd)** of ordinary points as a quotient of two such ratios for **a** and **b.**

Proof. Choose homogeneous coordinates α, β, γ for **A, B, C** such that $\gamma = \alpha + \beta$. By the cross ratio assumptions, we know that homogeneous coordinates $\delta_m^{\,*}$ for $\mathbf{D}_m$ are given by $z_m\,\alpha + \beta$. A straightforward calculation shows that

$$(z_1 - z_2)\,\delta_3^{\,*} \;=\; (z_3 - z_2)\,\delta_1^{\,*} + (z_1 - z_3)\,\delta_2^{\,*}$$

and similarly we have

$$(z_1 - z_2)\,\delta_4^{\,*} \;=\; (z_4 - z_2)\,\delta_1^{\,*} + (z_1 - z_4)\,\delta_2^{\,*}.$$

Thus if $\delta_1 = (z_3 - z_2)\,\delta_1^{\,*}$ and $\delta_2 = (z_1 - z_3)\delta_2^{\,*}$, then we have

$$(z_1 - z_2)\,\delta_4^{*} \;=\; \frac{(z_4 - z_2)}{(z_3 - z_2)}\,\delta_1 + \frac{(z_1 - z_4)}{(z_1 - z_3)}\,\delta_2\,.$$

The cross ratio formula follows immediately from the equation above and the definition of the cross ratio.■

Proposition 4. *Let* $n = 2$ *or* 3, *and suppose that* **a, b, c** *and* **d** *are four collinear points in* $\mathbf{R}^n$ *such that* **c** *divides* **a** *and* **b** *in the ratio* $1 - t : t$ *and* **d** *divides* **a** *and* **b** *in the ratio* $1 - s : s$. *Then the cross ratio* **(abcd)** *is given by the following quotient*:

$$\frac{1-t}{t}\Big/\frac{1-s}{s}$$

Proof. Recall that the homogeneous coordinates for an ordinary point **x** in $\mathbf{R}^n$ are given by the $(n+1)$ – dimensional (column) vector ξ^* corresponding to $(\mathbf{x}, 1)$. Therefore, if we take α^*, β^*, and γ^* to be the homogeneous coordinates for **a, b,** and **c**

defined in this fashion then we have $\gamma^* = t\alpha^* + (1-t)\beta^*$; of course, if we define δ^* similarly with respect to **d,** then we also have $\delta^* = s\alpha^* + (1-s)\beta^*$. The preceding sentence shows that if we make new homogeneous coordinates with $\alpha = t\alpha^*$ and $\beta = (1-t)\beta^*$, then we have $\gamma^* = \alpha + \beta$. Furthermore, it follows that we may write $\delta^* = x\alpha + y\beta$ where $x = (1-t)/t$ and $y = (1-s)/s$. The formula for **(abcd)** follows immediately from these equations and the definition of the cross ratio.■

The following identity is also useful in many contexts.

__**Proposition 5.**__ *Let* **a, b, c** *be distinct collinear points in* $\mathbf{R}^n$ *with* **c** $= (1-t)$**b** $+ t$**a**, *and suppose that* **J** *is the ideal point on the extended projective line containing* **a, b,** *and* **c.** *Then* **(abcJ)** *is equal to* $(1-t)/t$.

__***Important special case.***__ In the notation of the proposition, we see that $t = ½$ if and only if **(abcJ)** $= -1$. More generally, an ordered set of four collinear points **W, X, Y, Z** is said to be a ***harmonic set*** if we have **(WXYZ)** $= -1$. By the previous result on the cross ratios for reorderings (or permutations) of the given four points, we know that if **(WXYZ)** $= -1$ then we also have

$$-1 = \mathbf{(WXYZ)} = \mathbf{(XWZY)} = \mathbf{(YZWX)} = \mathbf{(ZYXW)}$$

$$= \mathbf{(WXZY)} = \mathbf{(XWYZ)} = \mathbf{(ZYWX)} = \mathbf{(YZWX)}.$$

__***Proof of Proposition 5.***__ Define homogeneous coordinates for **a, b,** and **c** as in the proof of the preceding result. It follows that **J** has homogeneous coordinates given by

$$(\mathbf{b}-\mathbf{a}, 0) = \beta^* - \alpha^* = x\alpha + y\beta$$

where $y = 1/(1-t)$ and $x = (-1)/t$. The formula for **(abcJ)** follows immediately from this equation and the definition of the cross ratio.■

Perspective invariance of the cross ratio

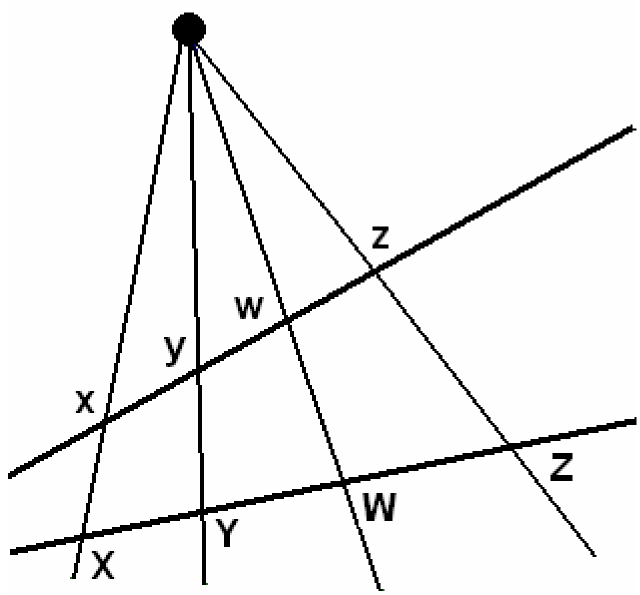
It is now time to prove that the cross ratio of four collinear points does not change under perspective projections. One way of doing so is to dualize the notion of cross ratio to lines in the projective plane and planes in projective $3-$space using homogeneous coordinates for such objects. We shall concentrate on the $2-$dimensional case and sketch the changes that are needed to handle everything in one higher dimension.
In the planar case, we start with four distinct concurrent lines $\mathbf{L}_1$, $\mathbf{L}_2$, $\mathbf{L}_3$, and $\mathbf{L}_4$. We then know that we can choose homogeneous coordinates λ_1, λ_2, λ_3 for the first three lines such that $\lambda_3 = \lambda_1 + \lambda_2$, and if we use the same vectors we can write homogeneous coordinates λ_4 for $\mathbf{L}_4$ in the form $u\lambda_1 + v\lambda_2$, where u and v are nonzero. The cross ratio $(\mathbf{L}_1\mathbf{L}_2\mathbf{L}_3\mathbf{L}_4)$ is then defined to be the quotient u/v exactly as before, and the previous reasoning shows that the value of this quotient does not depend upon the choices of λ_1, λ_2, λ_3 and λ_4.

Theorem 6. (Plane duality principle for cross ratios) *Let* $\mathbf{L}_1$, $\mathbf{L}_2$, $\mathbf{L}_3$, *and* $\mathbf{L}_4$ *be distinct concurrent lines in* $\mathbb{P}(\mathbb{R}^2)$, *and let* $\mathbf{M}$ *be another line which does not contain the point where the first four lines meet. Let* $\mathbf{A}_m$ *be the point at which* $\mathbf{M}$ *meets* $\mathbf{L}_m$, *where* $m = 1, 2, 3, 4$. *Then we have* $(\mathbf{L}_1\mathbf{L}_2\mathbf{L}_3\mathbf{L}_4) = (\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4)$.

Before proving result, we shall use it to show the perspective invariance of the cross ratio in the projective plane.

Theorem 7. (Perspective invariance of cross ratios) *Let* $\mathbf{L}_1$, $\mathbf{L}_2$, $\mathbf{L}_3$, *and* $\mathbf{L}_4$ *be distinct concurrent lines in* $\mathbb{P}(\mathbb{R}^2)$, *let* $\mathbf{M}$ *and* $\mathbf{N}$ *be distinct lines which do not contain the common point of the original four lines, and for* $m = 1, 2, 3, 4$ *take* $\mathbf{A}_m$ *and* $\mathbf{B}_m$ *to be the intersection points of* $\mathbf{L}_m$ *with* $\mathbf{M}$ *and* $\mathbf{N}$ *respectively. Then the cross ratios satisfy the equation* $(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4) = (\mathbf{B}_1\mathbf{B}_2\mathbf{B}_3\mathbf{B}_4)$.

In the drawing below, the preceding theorem implies that $(\mathbf{XYWZ}) = (\mathbf{xywz})$.



(Source: http://cellular.ci.ulsa.mx/comun/summer99/mcintosh/node3.html)

Proof of perspective invariance. Two applications of the previous theorem show that $(\mathbf{L}_1\mathbf{L}_2\mathbf{L}_3\mathbf{L}_4) = (\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4)$ and $(\mathbf{L}_1\mathbf{L}_2\mathbf{L}_3\mathbf{L}_4) = (\mathbf{B}_1\mathbf{B}_2\mathbf{B}_3\mathbf{B}_4)$. ■

Proof of cross ratio duality principle. Let $x = (\mathbf{A}_1\,\mathbf{A}_2\,\mathbf{A}_3\,\mathbf{A}_4)$ and $y = (\mathbf{L}_1\,\mathbf{L}_2\,\mathbf{L}_3\,\mathbf{L}_4)$. Choose homogeneous coordinates $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ for $\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \mathbf{L}_4$ and $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ for $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4$ such that $\lambda_3 = \lambda_1 + \lambda_2$ and $\alpha_3 = \alpha_1 + \alpha_2$. Then by construction we have $\lambda_4 = x\lambda_1 + \lambda_2$ and $\alpha_4 = y\alpha_1 + \alpha_2$. Since $\mathbf{A}_m \in \mathbf{L}_m$ for each m, we have $\lambda_m\alpha_m = 0$ for all m. In particular, these equations imply

$$0 = \lambda_3\alpha_3 = (\lambda_1 + \lambda_2)(\alpha_1 + \alpha_2) = \lambda_1\alpha_1 + \lambda_1\alpha_2 + \lambda_2\alpha_1 + \lambda_2\alpha_2 =$$

$$0 + \lambda_1\alpha_2 + \lambda_2\alpha_1 + 0 = \lambda_1\alpha_2 + \lambda_2\alpha_1$$

so that $\lambda_1\alpha_2 = -\lambda_2\alpha_1$; this number is nonzero because $\mathbf{A}_2$ does not lie on $\mathbf{L}_1$ and $\mathbf{A}_1$ does not lie on $\mathbf{L}_2$. Therefore we see that

$$\mathbf{0} = \lambda_4\alpha_4 = (x\,\lambda_1 + \lambda_2)\,(y\alpha_1 + \alpha_2) = y\lambda_2\,\alpha_1 + x\lambda_1\alpha_2 = (x-y)\lambda_1\alpha_2$$

and since $\lambda_1\,\alpha_2$ is nonzero it follows that $x - y = \mathbf{0}$, which means that $x = y$.■

The 3 – dimensional case. Regardless of whether we are working in the projective plane or projective 3 – space, we need to assume that the four concurrent lines lie in a single plane. There are several ways of doing this, and we shall choose one which reflects 3 – dimensional duality. In analogy with the 2 – dimensional case, if we are given four planes $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{Q}_3$, and $\mathbf{Q}_4$ which all contain a given line, then we may define the cross ratio $(\mathbf{Q}_1\ \mathbf{Q}_2\ \mathbf{Q}_3\ \mathbf{Q}_4)$ using homogeneous coordinates, and one has a duality principle for cross ratios which is analogous to the one presented above:

Theorem 8. (3 – dimensional duality principle for cross ratios) *Let* $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{Q}_3$, *and* $\mathbf{Q}_4$ *be distinct planes which all contain a single line in* $\mathbb{P}(\mathbf{R}^3)$, *and let* $\mathbf{N}$ *be a line which does not contain the line common to the first four planes and is not contained in any of the original four planes. Let* $\mathbf{A}_m$ *be the point where* $\mathbf{N}$ *meets* $\mathbf{Q}_m$ *for* $m = 1, 2, 3, 4$. *Then we have* $(\mathbf{Q}_1\ \mathbf{Q}_2\ \mathbf{Q}_3\ \mathbf{Q}_4) = (\mathbf{A}_1\ \mathbf{A}_2\ \mathbf{A}_3\ \mathbf{A}_4)$.

The proof is basically the same as in the 2 – dimensional case, the only difference being that we are working with homogeneous coordinates in $\mathbf{R}^4$ rather than $\mathbf{R}^3$.■

Theorem 9. (3 – dimensional perspective invariance of cross ratios) *Let* $\mathbf{L}_1$, $\mathbf{L}_2$, $\mathbf{L}_3$, *and* $\mathbf{L}_4$ *be distinct concurrent lines which all lie in some plane* $\mathbf{S}$ in $\mathbb{P}(\mathbf{R}^3)$, *let* $\mathbf{M}$ *and* $\mathbf{N}$ be *distinct lines in* $\mathbf{S}$ *which do not contain the common point of the original four lines, and* for $m = 1, 2, 3, 4$ *take* $\mathbf{A}_m$ *and* $\mathbf{B}_m$ *to be the intersection points of* $\mathbf{L}_m$ with $\mathbf{M}$ *and* $\mathbf{N}$ respectively. Then $(\mathbf{A}_1\ \mathbf{A}_2\ \mathbf{A}_3\ \mathbf{A}_4) = (\mathbf{B}_1\ \mathbf{B}_2\ \mathbf{B}_3\ \mathbf{B}_4)$.

Sketch of proof. In order to apply the preceding theorem, we need to find four planes $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{Q}_3$, and $\mathbf{Q}_4$ which all contain some auxiliary line and are somehow related to the lines $\mathbf{L}_1$, $\mathbf{L}_2$, $\mathbf{L}_3$, *and* $\mathbf{L}_4$. Let $\mathbf{X}$ be the point on $\mathbf{S}$ where the four lines meet, and let $\mathbf{Y}$ be a point which does not lie in $\mathbf{S}$. For each index value m take $\mathbf{Q}_m$ to be the unique plane containing the line $\mathbf{L}_m$ and the point $\mathbf{Y}$. It then follows that the planes $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{Q}_3$, and $\mathbf{Q}_4$ all contain the line $\mathbf{XY}$. By construction we also know that $\mathbf{Q}_m \cap \mathbf{S} = \mathbf{L}_m$, and hence it follows that the planes $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{Q}_3$, and $\mathbf{Q}_4$ must be distinct. We know that $\mathbf{N}$ is not equal to the common line $\mathbf{XY}$ of the planes $\mathbf{Q}_m$ because it does not contain the point $\mathbf{X}$; we must also check that $\mathbf{N}$ is not contained in any of the planes $\mathbf{Q}_m$. If this were so, then $\mathbf{L}$ would be contained in $\mathbf{Q}_m \cap \mathbf{S} = \mathbf{L}_m$, and since we know $\mathbf{N} \neq \mathbf{L}_m$ for all m it follows that $\mathbf{N}$ is not contained in any of the four planes we constructed. Similar considerations show that $\mathbf{M}$ is not equal to $\mathbf{XY}$ and is not contained in any of the planes $\mathbf{Q}_m$.

By construction, the lines $\mathbf{M}$ and $\mathbf{N}$ meet the planes $\mathbf{Q}_m$ in the points $\mathbf{A}_m$ and $\mathbf{B}_m$ respectively. Therefore the 3 – dimensional duality principle for cross ratios implies that $(\mathbf{Q}_1\ \mathbf{Q}_2\ \mathbf{Q}_3\ \mathbf{Q}_4) = (\mathbf{A}_1\ \mathbf{A}_2\ \mathbf{A}_3\ \mathbf{A}_4)$ and $(\mathbf{Q}_1\ \mathbf{Q}_2\ \mathbf{Q}_3\ \mathbf{Q}_4) = (\mathbf{B}_1\ \mathbf{B}_2\ \mathbf{B}_3\ \mathbf{B}_4)$, which immediately yield the desired relationship $(\mathbf{A}_1\ \mathbf{A}_2\ \mathbf{A}_3\ \mathbf{A}_4) = (\mathbf{B}_1\ \mathbf{B}_2\ \mathbf{B}_3\ \mathbf{B}_4)$.■

Applications to making measurements. Many textbooks on elementary Euclidean geometry contain discussions or exercises which indicate how one can use standard facts of Euclidean geometry to find distances or angle measurements when it is not possible to do by some direct means such as a ruler or protractor. The theorems on perspective invariance of cross ratios can also be used in some situations to find the distance between two points indirectly from a photograph. In the cross ratio drawing given above, suppose that the line whose points are denoted by small letters is on the picture and the other line is the one which has been photographed. Then we can measure the distances between all the points **x, y, z, w** on the picture and use them to compute the cross ratio **(xywz).** By the theorems on perspective invariance, we know this is also the cross ratio **(XYWZ);** often we may know the distances between three of the four points for some reason, and if we do then we can use the equality of the cross ratios to find the distances between all of the four points. Examples are discussed in the exercises.

Projective collineations

In this unit we have constructed projective extensions of the plane and $3-$space. Our next objective is to explain how one can construct projective extensions of affine transformation defined for $\mathbf{R}^2$ and $\mathbf{R}^3$ to well – behaved transformations for $\mathbb{P}(\mathbf{R}^2)$ and $\mathbb{P}(\mathbf{R}^3)$. It will be convenient to begin by generalizing the abstract notion of collineation to projective spaces.

Definition. Let $n = 2$ or 3. A ***projective collineation*** of $\mathbb{P}(\mathbf{R}^n)$ is a $1-1$ onto mapping Φ from $\mathbb{P}(\mathbf{R}^n)$ to itself such that the following hold:

1. The mapping Φ sends collinear sets to collinear sets and noncollinear sets to noncollinear sets.
2. If $n = 3$, the mapping Φ also sends coplanar sets to coplanar sets and noncoplanar sets to noncoplanar sets.

Frequently it is convenient to have a weaker criterion for recognizing projective collineations; a reader who wishes to skip the proof of this characterization may do so because the details of the argument will not be cited at any later point.

Proposition 10. *Let $n = 2$ or 3, and let Φ be a $1-1$ onto mapping Φ from $\mathbb{P}(\mathbf{R}^n)$ to itself. Then Φ is a projective collineation if and only if the following hold*:

(1) *For each subset of three distinct points* **{X, Y, Z}** *in* $\mathbb{P}(\mathbf{R}^n)$, *the points* **X, Y,** *and* **Z** *are collinear if and only if their images* **Φ(X), Φ(Y),** *and* **Φ(Z)** *are collinear.*

(2) [*Only applicable if* $n = 3$] *For each subset of four noncollinear points in* $\mathbb{P}(\mathbf{R}^n)$, *the points* **W, X, Y,** *and* **Z** *are coplanar if and only if their images* **Φ(W), Φ(X), Φ(Y),** *and* **Φ(Z)** *are coplanar.*

Proof. By definition a projective collineation automatically satisfies the conditions in the theorem, so the real work in front of us is to prove that the two conditions imply that the map Φ is a projective collineation.

Let $\mathcal{E}$ be a subset of $\mathbb{P}(\mathbf{R}^n)$, and let $\Phi(\mathcal{E})$ denote the set of points expressible as **Φ(X)** for $\mathbf{X} \in \mathcal{E}$. Since two point sets are automatically collinear, we might as well assume that $\mathcal{E}$ has at least three points. Let **X** and **Y** be two points in $\mathcal{E}$. If the latter are collinear, then every other point $\mathbf{Z} \in \mathcal{E}$ will also lie on **XY,** and thus by the first condition in the proposition we know that **F(Z)** will lie on the line joining Φ**(X)** and Φ**(Y)** and hence $\Phi(\mathcal{E})$ will be collinear. On the other hand, if some points $\mathbf{Z} \in \mathcal{E}$ does not lie on **XY,** then we know that Φ**(Z)** does not lie on the line joining Φ**(X)** and Φ**(Y),** so the set $\Phi(\mathcal{E})$ will not be collinear. This completes the proof of the first statement in the theorem.

Suppose now that $n = 3$; we need to prove the second statement of the theorem in that case. The ideas are similar to those of the previous paragraph. Let $\mathcal{E}$ and $\Phi(\mathcal{E})$ be as before; since sets with two or three points are automatically coplanar, we might as well assume that $\mathcal{E}$ has at least four points. Let **W, X,** and **Y** be three noncollinear points in $\mathcal{E}$; we are assuming that e is not collinear, so one can find such a triple of points. If the set $\mathcal{E}$ is coplanar, then every other point $\mathbf{Z} \in \mathcal{E}$ will also lie in the plane **WXY,** and thus by the first condition in the proposition we know that **F(Z)** will lie on the plane containing joining Φ**(W),** Φ**(X),** and Φ**(Y)** and hence $\Phi(\mathcal{E})$ will be coplanar. On the other hand, if some point $\mathbf{Z} \in \mathcal{E}$ does not lie in the plane **WXY,** then we know that Φ**(Z)** does not lie in the line joining Φ**(W)**Φ**(X)**Φ**(Y),** so the set $\Phi(\mathcal{E})$ will not be coplanar. This completes the proof of the second statement in the theorem.■

The usual sorts of arguments now yield analogs of some simple results about isometries, similarities, and affine transformations.

__**Proposition 11.**__ *Let $n = 2$ or 3. The identity map is a projective collineation from $\mathbb{P}(\mathbf{R}^n)$ to itself. If* **T** *is a projective collineation from $\mathbb{P}(\mathbf{R}^n)$ to itself, then so is its inverse* $\mathbf{T}^{-1}$. *Finally, if* **T** *and* **U** *are projective collineations from $\mathbb{P}(\mathbf{R}^n)$ to itself, then so is their composite* $\mathbf{T} \circ \mathbf{U}$.

Such abstract principles are important, but we also need to find a method for constructing nontrivial projective collineations. The next result does this for us.

__**Theorem 12.**__ *Let $n = 2$ or 3, and let* **A** *be an invertible $(n+1) \times (n+1)$ matrix with real entries. Then there is an associated projective collineation $\Phi_{\mathbf{A}}$ such that for each point* **X**, *if ξ is a set of homogeneous coordinates for* **X** *then* $\mathbf{A}\xi$ *is a set of homogeneous coordinates for $\Phi_{\mathbf{A}}(\mathbf{X})$. Furthermore, the construction sending* **A** *to $\Phi_{\mathbf{A}}$ has the following properties*:

1. *If $\mathbf{I}$ denotes the identity map of $\mathbf{R}^n$, then $\Phi_{\mathbf{I}}$ is the identity map of $\mathbb{P}(\mathbf{R}^n)$.*
2. *For all* **A** *and* **B**, *we have* $\Phi_{\mathbf{AB}} = \Phi_{\mathbf{A}} \circ \Phi_{\mathbf{B}}$.
3. *If* $\mathbf{B} = \mathbf{A}^{-1}$, *then* $\Phi_{\mathbf{B}} = (\Phi_{\mathbf{A}})^{-1}$.

The projective collineations $\Phi_{\mathbf{A}}$ are said to be ***algebraically specified.*** Since there are many invertible matrices **A** which take some nonzero vector **x** to a vector **Ax** which is not a scalar multiple of **x,** it is clear that there are algebraically specified projective collineations other than the identity. In fact, we shall verify below that every affine transformation of $\mathbf{R}^n$ defines an algebraically specified projective collineation of $\mathbb{P}(\mathbf{R}^n)$. In fact, one of the exercises for this section proves the following:

If **A** *and* **B** *are invertible* $(n+1)\times(n+1)$ *matrices with real entries that are not (nonzero) scalar multiples of each other, then* $\Phi_{\mathbf{A}}$ *and* $\Phi_{\mathbf{B}}$ *define distinct projective collineations of* $\mathbb{P}(\mathbb{R}^n)$.■

Since two vectors which are nonzero scalar multiples of each other always define the same point, we know that, conversely, $\Phi_{\mathbf{A}} = \Phi_{\mathbf{B}}$ if the two invertible matrices **A** and **B** are nonzero scalar multiples of each other.

Proof of Theorem **12.** The first step is to show that the construction described in the statement of the theorem is well – defined; in other words, if **x** is a set of homogeneous coordinates for a point then so is **Ax,** and if **u** and **v** represent the same point **X,** then **Au** and **Av** are both homogeneous coordinates for points and in fact they represent the same point. Since **x** is a set of homogeneous coordinates for **X,** it is nonzero, and since **A** is invertible we also know that **Ax** is nonzero, so that it defines a point in $\mathbb{P}(\mathbb{R}^n)$. Furthermore, since **u** and **v** represent the same point, then $\mathbf{v} = c\,\mathbf{u}$ for some nonzero scalar c and thus by linearity we have $\mathbf{Av} = c\,\mathbf{Au}$, which shows that **Av** and **Au** define the same point in $\mathbb{P}(\mathbb{R}^n)$. This proves the existence of a mapping $\Phi_{\mathbf{A}}$ such that for each point **X,** if ξ is a set of homogeneous coordinates for **X** then $\mathbf{A}\xi$ is a set of homogeneous coordinates for $\Phi_{\mathbf{A}}(\mathbf{X})$.

Next, we need to show that $\Phi_{\mathbf{A}}$ is $1-1$ and onto. To see that it is $1-1$, observe first that if $\Phi_{\mathbf{A}}(\mathbf{X}) = \Phi_{\mathbf{A}}(\mathbf{Y})$ and **X** and **Y** have homogeneous coordinates given by ξ and η respectively, then we must have $\mathbf{A}\xi = c\,\mathbf{A}\eta$ for some nonzero scalar c. Linearity then implies $\mathbf{Au} = \mathbf{A}(c\,\eta)$ and since an invertible matrix defines a $1-1$ and onto mapping it follows that $\xi = c\,\eta$. Therefore $\Phi_{\mathbf{A}}$ is $1-1$; to see it is onto, let $\mathbf{Y} \in \mathbb{P}(\mathbb{R}^n)$, and let η be a set of homogeneous coordinates for **Y.** The invertibility of **A** implies that $\eta = \mathbf{A}\xi$ for some ξ, and since η is nonzero we know that ξ must also be nonzero. By construction, if ξ represents **X** we then have $\Phi_{\mathbf{A}}(\mathbf{X}) = \mathbf{Y}$.

Finally, we need to show the conditions involving sets of 3 and 4 points. Suppose first that **X, Y, Z** are distinct points in $\mathbb{P}(\mathbb{R}^n)$, and let $\xi,\ \eta,\ \zeta$ be homogeneous coordinates for these respective points. Then by construction the points **X, Y, Z** are noncollinear if and only if the vectors $\xi,\ \eta,\ \zeta$ do not span a subspace of dimension less than or equal to 2, and the latter holds if and only if $\xi,\ \eta,\ \zeta$ are linearly independent. Since invertible linear transformations send linearly independent points to linearly independent points and linearly dependent points to linearly dependent points, it follows that $\xi,\ \eta,\ \zeta$ are linearly independent if and only if $\mathbf{A}\xi,\ \mathbf{A}\eta,\ \mathbf{A}\zeta$ are linearly independent, and by the reasoning of the previous sentence this is true if and only if $\Phi_{\mathbf{A}}(\mathbf{X}), \Phi_{\mathbf{A}}(\mathbf{Y}), \Phi_{\mathbf{A}}(\mathbf{Z})$ are noncollinear. Combining these, we see that **X, Y, Z** are noncollinear if and only if the points $\Phi_{\mathbf{A}}(\mathbf{X}), \Phi_{\mathbf{A}}(\mathbf{Y}), \Phi_{\mathbf{A}}(\mathbf{Z})$ are. — Now suppose that **W, X, Y, Z** are distinct noncollinear points in $\mathbb{P}(\mathbb{R}^3)$, and let $\omega,\ \xi,\ \eta,\ \zeta$ be homogeneous coordinates for these respective points; since a set of points is collinear if and only if every subset consisting of exactly three members is collinear, the preceding discussion implies that the points $\Phi_{\mathbf{A}}(\mathbf{W}), \Phi_{\mathbf{A}}(\mathbf{X}), \Phi_{\mathbf{A}}(\mathbf{Y}), \Phi_{\mathbf{A}}(\mathbf{Z})$ are also noncollinear, and from here we can give an argument very similar to the previous one for three distinct points. Specifically, by construction the points **W, X, Y, Z** are noncoplanar if and only if the vectors ω, ξ, η, ζ do not span a subspace of dimension less than or equal to 3, and the latter holds if and only if ω, ξ, η, ζ are linearly independent. Since invertible linear transformations send linearly independent points to linearly independent points and linearly dependent points to linearly dependent points, it follows that ω, ξ, η, ζ are linearly independent if and only if $\mathbf{A}\omega$, $\mathbf{A}\xi$, $\mathbf{A}\eta$, $\mathbf{A}\zeta$ are linearly independent, and by the reasoning of the previous sentence this is true if and only if $\Phi_{\mathbf{A}}(\mathbf{W}), \Phi_{\mathbf{A}}(\mathbf{X}), \Phi_{\mathbf{A}}(\mathbf{Y}), \Phi_{\mathbf{A}}(\mathbf{Z})$ are noncoplanar. Combining these, we see that the points **W, X, Y, Z** are noncoplanar if and only if their image points $\Phi_{\mathbf{A}}(\mathbf{W}), \Phi_{\mathbf{A}}(\mathbf{X}), \Phi_{\mathbf{A}}(\mathbf{Y}), \Phi_{\mathbf{A}}(\mathbf{Z})$ are.■

Notational convention. Given the standard equivalence between linear transformations from $\mathbf{R}^{n+1}$ to itself and the set of $(n+1) \times (n+1)$ matrices (in which the matrix determines a linear transformation by left multiplication), it is sometimes useful to use similar terminology if we are given an invertible linear transformation from $\mathbf{R}^{n+1}$ to itself; specifically, if **T** is such a linear transformation, then $\Phi_{\mathbf{T}}$ will denote the associated projective collineation on $\mathbb{P}(\mathbf{R}^n)$ characterized by the sort of relationship described in the theorem: If the nonzero vector ξ represents the point **X,** then $\Phi_{\mathbf{T}}(\mathbf{X})$ is represented by $\mathbf{T}(\xi)$.

Projectivization of affine transformations. We have already noted that an affine transformations of $\mathbf{R}^n$ extends to a projective collineation of $\mathbb{P}(\mathbf{R}^n)$. Here is the formal statement of that result.

Theorem 13. *If* **T** *is the affine transformation on* $\mathbf{R}^n$ *given by* $\mathbf{T}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$, *where* **A** *is an invertible* $n \times n$ *matrix and* **b** *is a vector in* $\mathbf{R}^n$, *then the* $(n+1) \times (n+1)$ *matrix*

$$\Omega_{\mathbf{T}} = \begin{pmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{0} & 1 \end{pmatrix}$$

defines an extension of **T** *to an algebraically specified projective collineation* $\Psi_{\mathbf{T}}$ *of* $\mathbb{P}(\mathbf{R}^n)$.

Proof. Let **x** be a vector in $\mathbf{R}^n$, and let $\xi = (\mathbf{x}, 1)$ give the standard homogeneous coordinates for the ordinary point in $\mathbb{P}(\mathbf{R}^n)$ given by **x.** Then the block multiplication identity

$$\Omega_{\mathbf{T}}(\xi) = \begin{pmatrix} \mathbf{A} & \mathbf{b} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{A}\mathbf{x} + \mathbf{b} \\ 1 \end{pmatrix}$$

shows that $\Psi_{\mathbf{T}}$ maps the ordinary point **x** to the ordinary point $\mathbf{T}(\mathbf{x})$.■

Corollary 14. *The construction associating a projective collineation* $\Psi_{\mathbf{T}}$ *to an affine transformation* **T** *has the following properties:*

1. *If* $\mathbf{I}$ *denotes the identity transformation on* $\mathbf{R}^n$, *then* $\Psi_{\mathbf{I}}$ *is the identity map on* $\mathbb{P}(\mathbf{R}^n)$.
2. *For all affine transformations* **T** *and* **U,** *we have* $\Psi_{\mathbf{TU}} = \Psi_{\mathbf{T}} \circ \Psi_{\mathbf{U}}$.

3. *If* $\mathbf{S} = \mathbf{T}^{-1}$, *then* $\Psi_\mathbf{S} = (\Psi_\mathbf{T})^{-1}$.

Proof. Since $\Psi_\mathbf{T} \circ \Phi\Omega(\mathbf{T})$, where $\Omega(\mathbf{T})$ is the matrix $\Omega_\mathbf{T}$ described above (the notation is rewritten to avoid subscripts of subscripts), by the group theorem for projective collineations it suffices to prove that $\Omega(\mathbf{I})$ is the identity mapping, $\Omega(\mathbf{T}\circ\mathbf{U}) = \Omega(\mathbf{T})\Omega(\mathbf{U})$, and $\Omega(\mathbf{T}^{-1}) = [\Omega(\mathbf{T})]^{-1}$. The first statement follows immediately from the description of $\Omega(\mathbf{I})$ given in the statement of the proposition, so we can focus our attention on the remaining two assertions.

At this point it is helpful to review some observations about affine transformations from the discussion of the latter in Section **II.4.** First, if $\mathbf{T(x)} = \mathbf{Ax + b}$ where $\mathbf{A}$ is invertible and $\mathbf{b}$ is some vector, then the inverse is given by $\mathbf{T}^{-1}\mathbf{(y)} = \mathbf{A}^{-1}\mathbf{y} - \mathbf{A}^{-1}\mathbf{b}$, and using block multiplication of matrices one can check directly from this equation that $\Omega(\mathbf{T}^{-1}) = [\Omega(\mathbf{T})]^{-1}$. Second, if $\mathbf{U}$ is also an affine transformation and $\mathbf{U(x)} = \mathbf{Cx + d}$, where once again $\mathbf{C}$ is invertible, then we have $\mathbf{T}\circ\mathbf{U(x)} = \mathbf{AC(x)} + (\mathbf{Ab + d})$, and again using block multiplication of matrices one can verify $\Omega(\mathbf{TU}) = \Omega(\mathbf{T})\,\Omega(\mathbf{U})$ directly from the formula for the composite.■

Examples. To see that ***not every projective collineation comes from an affine transformation,*** consider the permutation matrix $\mathbf{A}$ whose columns (from left to right) are the permuted unit vectors $\mathbf{e}_2, \ldots, \mathbf{e}_{n+1}, \mathbf{e}_1$. Probably the easiest way to see that $\Phi_\mathbf{A}$ is not equal to $\Psi_\mathbf{T}$ for any affine transformation $\mathbf{T}$ is that each $\Psi_\mathbf{T}$ takes the ideal line or plane defined by $x_{n+1} = 0$ into itself, and the mapping $\Phi_\mathbf{A}$ takes it to the line or plane defined by $x_n = 0$.

Fundamental Theorem of Projective Geometry

In Units **II** and **III** (particularly Sections **II.4** and **III.5**) we described three basic types of geometric transformations, and we also showed that substantial families of such transformations could be specified in algebraic terms, and in this section we described yet another type of geometric transformation (projective collineations) which can be viewed as containing all the others as special cases. The table below lists the families considered in these notes; in each row the transformations are more general than the previous one, with the abstract synthetic geometric description in the first column and the key algebraically definable subfamilies in the second.

Geometric transformation type	**Defined on**	**Algebraically specified examples**
(Abstract) isometries	$\mathbf{R}^n$	Galilean transformations
Abstract similarity transformations	$\mathbf{R}^n$	(Algebraically specified) similarity transformations
(Abstract) affine collineations	$\mathbf{R}^n$	Affine transformations (algebraically specified)
(Abstract) projective collineations	$\mathbb{P}(\mathbf{R}^n)$	Algebraically defined projective collineations

In each rows except the last, we have mentioned that all geometric transformations described in the first column are given by the algebraically defined transformations in the second. One key part of the Fundamental Theorem of Projective Geometry states that the same also holds for projective collineations of $\mathbb{P}(\mathbb{R}^n)$, where $n = 2$ or 3.

Theorem 15. (Fundamental Theorem of Projective Geometry) *Let $n = 2$ or 3, and assume that the following hold:*

1. *If $n = 2$, then $\{X_1, \ldots, X_4\}$ and $\{Y_1, \ldots, Y_4\}$ are sets of distinct points, no three of which are collinear.*
2. *If $n = 3$, then $\{X_1, \ldots, X_5\}$ and $\{Y_1, \ldots, Y_5\}$ are sets of are distinct points, no four of which are coplanar.*

Then there is a unique projective collineation Φ from $\mathbb{P}(\mathbb{R}^n)$ to itself such that $\Phi(X_k) = Y_k$ for $k = 1, \ldots, n+2$.

We shall prove the existence half of this theorem; this will be done by algebraic methods. In contrast, the uniqueness part involves synthetic methods, and the proof requires a considerable amount of algebraic and geometric input that is beyond the scope of this course. Here is one reference for the proof:

> R. Bumcrot, ***Modern Projective Geometry.*** Holt, Rinehart and Winston, New York, 1969. ASIN: B–000–6BYLO–4.

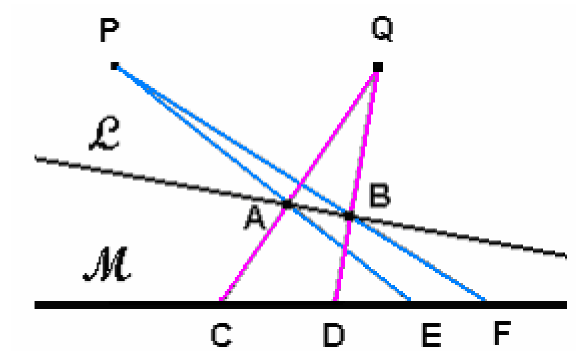
Proof of existence in the Fundamental Theorem. By the results from Section 5 on choosing homogeneous coordinates, there are homogeneous coordinates $\xi_1, \ldots, \xi_{n+2}$ and $\eta_1, \ldots, \eta_{n+2}$ for the points $X_1, \ldots, X_{n+2}$ and $Y_1, \ldots, Y_{n+2}$ (respectively) such that $\xi_{n+2} = \xi_1 + \ldots + \xi_{n+1}$ and $\eta_{n+2} = \eta_1 + \ldots + \eta_{n+1}$, and we also know that the vectors $\xi_1, \ldots, \xi_{n+1}$ and $\eta_1, \ldots, \eta_{n+1}$ give bases for $\mathbb{R}^{n+1}$. Therefore there is an invertible linear transformation $\mathbf{T}$ of $\mathbb{R}^{n+1}$ such that $\mathbf{T}(\xi_k) = \eta_k$ for $k = 1, \ldots, n+1$; by fundamental results from linear algebra we know that $\mathbf{T}$ is given by some invertible matrix $\mathbf{A}$. By the linearity of matrix multiplication, we also have $\mathbf{A}(\xi_{n+2}) = \eta_{n+2}$, and therefore we have $\Phi_{\mathbf{A}}(X_k) = Y_k$ for $k = 1, \ldots, n+2$.■

The following companion to the Fundamental Theorem of Projective Geometry is extremely important for many purposes.

Proposition 16. (Complement to the Fundamental Theorem) *Let $n = 2$ or 3, and let $\mathbf{T}$ be a projective collineation of $\mathbb{P}(\mathbb{R}^n)$. If W, X, Y, Z are distinct collinear points in $\mathbb{P}(\mathbb{R}^n)$, then so are their images, and we have $(WXYZ) = (T(W)\,T(X)\,T(Y)\,T(Z))$.*

Proof of Proposition 16 for algebraically specified projective collineations. By the Fundamental Theorem we know that $\mathbf{T} = \Phi_{\mathbf{A}}$ for some invertible matrix $\mathbf{A}$. Let ω, ξ, η, ζ be homogeneous coordinates for W, X, Y, Z such that $\eta = \omega + \xi$; then we have $\zeta = u\,\omega + v\,\xi$, where the coefficients satisfy $(WXYZ) = u/v$. By the construction of $\mathbf{T} = \Phi_{\mathbf{A}}$ it follows that $\mathbf{A}\omega$, $\mathbf{A}\xi$, $\mathbf{A}\eta$, $\mathbf{A}\zeta$ are homogeneous coordinates for $T(W)$, $T(X)$, $T(Y)$, $T(Z)$. The linearity of $\mathbf{A}$ implies that $\mathbf{A}\eta = \mathbf{A}\omega + \mathbf{A}\xi$ as well as $\mathbf{A}\zeta = u\,\mathbf{A}\omega + v\,\mathbf{A}\xi$, and thus by the definition of cross ratio we conclude that $(WXYZ) = u/v$ is equal to the corresponding cross ratio $(T(W)\,T(X)\,T(Y)\,T(Z))$.■

We shall conclude this unit by indicating how one can use projective collineations to obtain some insight into Alberti's question as formulated at the beginning of this section; recall this concerns the common properties shared by different perspective projections of the same object in space. We shall only look at a very simple problem of this type, in which one has two fixed planes, say $\mathcal{F}$ and $\mathcal{G}$, and we project from $\mathcal{F}$ to $\mathcal{G}$ using two distinct focal points which do not lie on either plane. The drawing below depicts a $2-$dimensional analog in which $\mathcal{F}$ and $\mathcal{G}$ are replaced by two lines $\mathcal{L}$ and $\mathcal{M}$; the points **P** and **Q** represent the two focal points. This figure turns out to be entirely adequate for analyzing the $3-$dimensional case.



Convention. In the computations below we identify a 1×1 matrix with its unique scalar entry.

In analogy with the drawing above, let $\mathcal{F}$ be the "object plane" and let $\mathcal{G}$ be the "image plane" for the perspective projection, and let **N** be the line in which they intersect; let **W** be the $3-$dimensional subspace of vectors in $\mathbf{R}^4$ whose homogeneous coordinates are representatives for the points of $\mathcal{G}$, and let **V** be the $3-$dimensional subspace of vectors in $\mathbf{R}^4$ whose homogeneous coordinates are representatives for the points of $\mathcal{G}$. The points **C** and **D** will represent projection centers for the perspective projections onto $\mathcal{G}$. By construction, neither **C** nor **D** lies on either of the planes $\mathcal{F}$ or $\mathcal{G}$, and thus for all sets of homogeneous coordinates γ', δ', ϕ' for **C, D,** $\mathcal{F}$ we know that $\phi'\gamma'$ and $\phi'\delta'$ are nonzero. There are nonzero multiples ϕ and γ of ϕ' and γ' such that $\phi\gamma = 1$, and we choose our homogeneous coordinates for **C** and $\mathcal{F}$ such that this equation is satisfied.

Every vector in $\mathbf{R}^4$ can be expressed (in fact, uniquely) as a sum $\alpha + q\gamma$ where $\alpha \in$ **W** and q is a scalar. If **D** is the second projection point as above, then we know that **D** does not lie on the plane $\mathcal{G}$, and therefore it follows that $\delta = \alpha + q\gamma$ where q must be nonzero. Dividing by q, we see that homogeneous coordinate δ for **D** can be chosen such that $\delta = \gamma + \beta$, where β lies in **W**. Note also that under these conditions every vector in $\mathbf{R}^4$ may also be expressed (uniquely) as a sum $\alpha' + p(\gamma + \beta)$, where $\alpha' \in$ **W** and p is a scalar.

We would like to define invertible linear transformations **S** and **T** on $\mathbf{R}^4$ such that for each point **X** on $\mathcal{G}$, the associated projective collineation $\Phi_{\mathbf{S}}$ maps **X** to the intersection point of the line **DX** and the plane $\mathcal{F}$, and similarly the associated projective collineation $\Phi_{\mathbf{T}}$ maps **X** to the intersection point of the line **CX** and the plane $\mathcal{F}$. Explicit formulas for the values of such linear transformations on a typical vectors $\alpha + k\gamma = \alpha' + m(\gamma + \beta)$ are given as follows:

$$\mathbf{S}(\alpha' + m(\gamma+\beta)) \;=\; (1 + \phi\beta)\,\alpha' - (\phi\alpha')\,(\gamma+\beta) + m\,(\gamma+\beta).$$

$$\mathbf{T}(\alpha + k\gamma) \;=\; \alpha + (k - \phi\alpha)\,\gamma$$

It follows that for all $\alpha \in \mathbf{W}$ we have

$$\mathbf{T}^{-1}\mathbf{S}(\alpha) \;=\; (1 + \phi\beta)\alpha - (\phi\alpha)\,\beta.$$

Suppose now that we take a basis $\alpha_1, \alpha_2, \alpha_3$ for **W** such that α_2 and α_3 form a basis for $\mathbf{V} \cap \mathbf{W}$ (and thus we also know that α_1 does not lie in **W**), and write the previously defined vector $\beta \in \mathbf{W}$ as a linear combination $x\,\alpha_1 + y\,\alpha_2 + z\,\alpha_3$. Replacing α_1 by a nonzero scalar multiple if necessary, we may assume that $\phi\alpha_1 = 1$, and if we do so then we must also have $x \neq -1$ in the expansion of β. If we apply the formula above for $\mathbf{T}^{-1}\mathbf{S}$ to the basis $\alpha_1, \alpha_2, \alpha_3$ described above, we have the following equations:

$$\mathbf{T}^{-1}\mathbf{S}(\alpha_1) \;=\; \alpha_1 - y\alpha_2 - z\alpha_3$$

$$\mathbf{T}^{-1}\mathbf{S}(\alpha_2) \;=\; (1+x)\,\alpha_2$$

$$\mathbf{T}^{-1}\mathbf{S}(\alpha_3) \;=\; (1+x)\,\alpha_3$$

The preceding formulas show that $\mathbf{T}^{-1}\mathbf{S}$ is an invertible linear transformation on **W**, and with respect to the ordered basis $\{\alpha_1, \alpha_2, \alpha_3\}$, the matrix of this linear transformation is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ -y & 1+x & 0 \\ -z & 0 & 1+x \end{pmatrix}.$$

If we multiply this matrix by the nonzero quantity $1/(1+x)$, then the new matrix defines the same mapping on the plane $\mathcal{G}$, and if we make the change of variables

$$a \;=\; \frac{1}{1+x}, \quad b \;=\; \frac{-y}{1+x}, \quad c \;=\; \frac{-z}{1+x}$$

then the new matrix takes the form

$$\begin{pmatrix} a & 0 & 0 \\ b & 1 & 0 \\ c & 0 & 1 \end{pmatrix}$$

where a, b, c depend upon the choice of **D** and for all such choices we have $a \neq 0$; in fact, with different choices of **D** we can realize all such matrices, where a is an arbitrary nonzero number and b, c are arbitrary real numbers.

Up to multiplication by a nonzero scalar, this form is nearly equivalent to the matrices described in Theorem **17** on page 129 of Ryan, the main difference being that the matrix in Ryan has only one possibly nonzero term off the main diagonal and ours may have two. However, if we have two nonzero terms off the diagonal, then we can replace our basis α_2 and α_3 for $\mathbf{V} \cap \mathbf{W}$ by another basis α_2', α_3' such that the matrix with respect to the new basis $\alpha_1, \alpha_2', \alpha_3'$ has at most one nonzero term off the diagonal.

Notation. The mappings from $\mathcal{G}$ to itself that are defined as above are called change of perspective transformations relative to $\mathcal{F}$. If $\mathcal{G}$ is the plane in $\mathbb{P}(\mathbf{R}^3)$ consisting of all points whose first homogeneous coordinate is equal to zero, then there is an obvious identification of $\mathcal{G}$ with $\mathbb{P}(\mathbf{R}^2)$, and it follows immediately that ***these change of perspective transformations correspond to projective collineations of*** $\mathbb{P}(\mathbf{R}^2)$.

Change of object plane

By the preceding discussion, if one changes the central focus point of a perspective projection from an object plane $\mathcal{F}$ to an image plane $\mathcal{G}$, then the image of points under the first projection are mapped to the image of points under the second by special types of projective collineations. Thus the common properties of the perspective projections considered above are basically the geometric properties of the projective plane that do not change under the special class of projective collineations described above.

More generally, one can also ask about common properties of different perspective projections onto $\mathcal{G}$ if one moves the object plane $\mathcal{F}$ to some other location, say $\mathcal{H}$, but keeps the image plane fixed. In some sense this is the reverse of the "real life" situation in which one keeps the object fixed but can move the central focus point and the image plane, but the reverse model provides a better mathematical setting in which to analyze the common properties of different perspective projections if one does not insist that the object and image are both held fixed.

Of course, if we let $\mathcal{F}$ vary, then the line of intersection $\mathcal{F} \cap \mathcal{G}$ will also vary over all the lines of $\mathcal{G}$. In particular, there are many ways of changing the plane $\mathcal{F}$ so that the line of intersection corresponds to the 2 – dimensional vector subspace spanned by α_1 and α_2 or by α_1 or α_3. If we do this, then we obtain 3×3 matrices like the preceding ones in which the rows and columns are rearranged by a permutation of $\{1, 2, 3\}$. Now the matrices displayed above include all matrices that are obtained from the identity by two types of elementary row operations:

1. Multiplying one row by a nonzero scalar.
2. Adding a multiple of one row to another.

There is a third type of elementary row operation (switching two rows), but it turns out that this operation can be expressed in terms of the other two.

Theorem 17. *Suppose that* **A** *is a square matrix obtained from the identity matrix by switching two rows. Then* **A** *can be obtained from the identity by finitely many row operations of the other two types.*

The proof is not difficult, but in order to avoid a detour in the main discussion we shall postpone the proof to the end of this section. If we combine the theorem on matrices with the previous discussion, we obtain the following result.

Theorem 18. *Let* **T** *be an algebraically specified projective collineation of* $\mathbf{P}(\mathbf{R}^2)$, *and view the latter as contained in* $\mathbf{P}(\mathbf{R}^3)$ as above. *Then* **T** *is expressible as a composite of change of perspective transformations.*

In particular, the preceding result implies that the common properties under general changes of perspective are the same as the geometrical properties of $\mathbf{P}(\mathbf{R}^2)$ that are left unchanged (in mathematical language, ***invariant***) under the entire family of projective collineations. We mention this because in some sense it closes the loop, showing that the projective geometry, which arose from the theory of perspective, feeds back to yield fundamental insights into that theory.

Proof of Theorem 18. By hypothesis the collineation **T** has the form $\Phi_{\mathbf{A}}$, where **A** is an invertible matrix. Since every invertible matrix is a product of elementary matrices and every elementary matrix is a product of elementary matrices of the first two types described above, we know that **A** is expressible as a product of elementary matrices, say $\mathbf{A} = \mathbf{E}_1 \dots \mathbf{E}_k$. If $\mathbf{T}_j$ is the algebraically specified projective collineation given by $\mathbf{T}_j$, then it follows from the formal properties of this construction that **T** is equal to the composite $\mathbf{T}_1 \circ \dots \circ \mathbf{T}_k$. By the preceding discussion, each of the collineations $\mathbf{T}_j$ is a change of perspective transformation, and thus we have expressed **T** as a composite of the desired type.■

Appendix – Proof of row reduction theorem

We claim it is enough to do the latter in the 2×2 case. In the general case, the matrix differs from the identity in only four entries; specifically, if one interchanges rows p and q where p and q, then the exceptional matrix entries are in positions (p, p), (p, q), (q, p) and (q, q). If we restrict ourselves to operations which only involve rows p and q, then each elementary matrix we obtain will have the same property, and hence if we can do the 2×2 case we can spread it out more generally by just applying the operations to rows p and q rather than rows 1 and 2.

Turning to the special case, we check that the matrix of the linear transformation of $\mathbf{R}^2$ which switches coordinates can be reduced to the identity by elementary row operations which either add a multiple of one row to another or multiply one row by a nonzero constant. The explicit steps are indicated by the sequence of matrices displayed below.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

For the sake of completeness, we shall list the elementary operations by which each matrix in this display is obtained from the preceding one:

1. Add the second row to the first.
2. Subract the first row from the second.
3. Add the second row to the first.
4. Multiply the second row by -1.

By the previous remarks, this completes the proof of the result on matrices.■