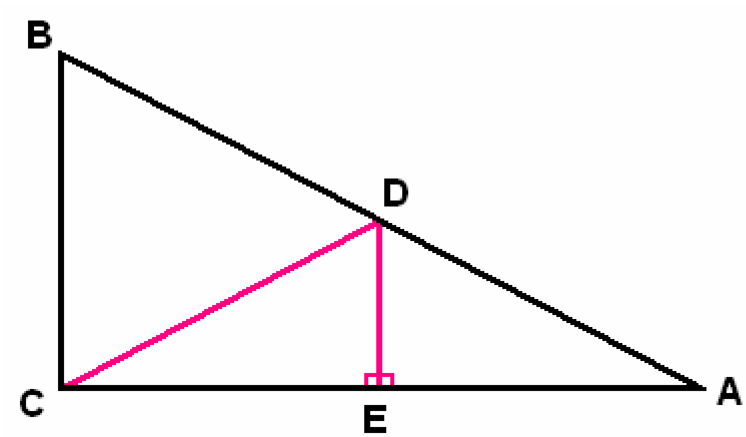
Solved exercises in neutral and hyperbolic geometry

Here are some further examples of problems similar to the exercises for Unit **V** with complete solutions.

PROBLEM 1. *Suppose that we are given a right triangle* $\triangle$**ABC** *in the hyperbolic plane* $\mathcal{P}$ *with a right angle at* **C**, *and let* **E** *denote the midpoint of* **[AB]**. *Prove that the line* **L** *perpendicular to* **AC** *through* **E** *contains a point* **D** *on* **(AB)** *and that* $d(\mathbf{B}, \mathbf{D})$ *is greater than* $d(\mathbf{A}, \mathbf{D}) = d(\mathbf{C}, \mathbf{D})$.



SOLUTION. First of all, by Pasch's Theorem we know that the perpendicular bisector **L** either contains a point of **[BC]** or of **(AB)**. However, since **AC** is perpendicular to both **BC** and **L** we know that the first option cannot happen, and therefore the line **L** must contain some point **D** of **(AB)**. By **SAS** we have $\triangle$**DEA** $\cong$ $\triangle$**DEC**, and therefore it follows that $d(\mathbf{A}, \mathbf{D}) = d(\mathbf{C}, \mathbf{D})$. Furthermore, we have $|\angle \mathbf{DAE}| = |\angle \mathbf{DCE}|$. By the additivity property for angle measurements, we have

$$|\angle \mathbf{DAE}| + |\angle \mathbf{DCB}| = |\angle \mathbf{DCE}| + |\angle \mathbf{DCB}| = 90^\circ$$

and if we combine this with $\angle \mathbf{DAE} = \angle \mathbf{BAC}$, $\angle \mathbf{CBD} = \angle \mathbf{CBA}$, and the hyperbolic angle – sum property

$$|\angle \mathbf{BAC}| + |\angle \mathbf{CDB}| < 90^\circ$$

we see that $|\angle \mathbf{DBC}| < |\angle \mathbf{CDB}|$. Since the larger angle is opposite the longer side, it follows that $d(\mathbf{B}, \mathbf{D}) > d(\mathbf{C}, \mathbf{D}) = d(\mathbf{A}, \mathbf{D})$.■

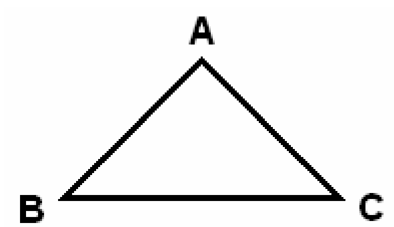
PROBLEM 2. *In the setting of the previous problem, determine whether* $|\angle \mathbf{BAC}|$ *is less than, equal to or greater than* $\frac{1}{2}\,|\angle \mathbf{BDC}|$.

SOLUTION. We have $|\angle \mathbf{ADC}| = |\angle \mathbf{CDE}| + |\angle \mathbf{EDA}|$ because the midpoint **E** lies in the interior of $\angle$**ADC**, and since $\triangle$**DEA** $\cong$ $\triangle$**DEC** it also follows that $|\angle \mathbf{ADC}| = 2\,|\angle \mathbf{EDA}|$. By the supplement property for angle measures we have $|\angle \mathbf{BDC}| + |\angle \mathbf{ADC}| = 180^\circ$. Therefore we also have $\frac{1}{2}\,|\angle \mathbf{BDC}| + |\angle \mathbf{EDA}| = 90^\circ$. On the other hand, hyperbolic angle – sum property implies that $|\angle \mathbf{BAC}| + |\angle \mathbf{EDA}| < 90°$. Therefore we have $|\angle \mathbf{BAC}| + |\angle \mathbf{EDA}| < ½|\angle \mathbf{BDC}| + |\angle \mathbf{EDA}|$, and if we subtract the second term from each side of this inequality we conclude that $|\angle \mathbf{EDA}| < ½|\angle \mathbf{BDC}|$.■

PROBLEM 3. *Suppose that we are given a right triangle* $\triangle$**ABC** *in the neutral plane* $\mathcal{P}$ *with a right angle at* **C,** *and let* **F** *denote the midpoint of* **[AB].** *Show that if* **F** *is equidistant from the vertices, then* $\mathcal{P}$ *is Euclidean.*

SOLUTION. If **F** is equidistant from the vertices, then **EF** is the perpendicular bisector of **[AC],** and hence we must have **F = D.** However, by the first problem we know **D** is ***not*** equidistant from the vertices if the plane $\mathcal{P}$ is hyperbolic, and therefore $\mathcal{P}$ must be Euclidean.■

PROBLEM 4. *Suppose that we are given an isosceles triangle* $\triangle$**ABC** *in the neutral plane* $\mathcal{P}$ *with* $d(\mathbf{A}, \mathbf{B}) = d(\mathbf{A}, \mathbf{C})$ *and* $|\angle \mathbf{BAC}| > 60°$. *Prove that* $d(\mathbf{B}, \mathbf{C}) > d(\mathbf{A}, \mathbf{C}) = d(\mathbf{A}, \mathbf{B})$.



Discussion. The drawing depicts an isosceles right triangle. As such, we know that its hypotenuse is longer than either of its legs, and this is in fact true in neutral geometry. The object of the exercise is to prove a more general result which is also true in neutral geometry.

SOLUTION. By the Saccheri – Legendre Theorem we have

$$|\angle \mathbf{BAC}| + |\angle \mathbf{ABC}| + |\angle \mathbf{ACB}| = |\angle \mathbf{BAC}| + 2\,|\angle \mathbf{ABC}| \leq 180°$$

and since $|\angle \mathbf{BAC}| > 60°$ it follows that $2\,|\angle \mathbf{ABC}| < 120°$ so that $|\angle \mathbf{ABC}| < 60°$. Since the larger angle of a triangle is opposite the longer side, we have $d(\mathbf{B}, \mathbf{C}) > d(\mathbf{A}, \mathbf{B})$, and the final part of the conclusion follows because the right hand side is equal to $d(\mathbf{A}, \mathbf{C})$.■

Note. In Euclidean geometry there is a companion result for isosceles triangles: *If* $|\angle \mathbf{BAC}| < 60°$, *then* $d(\mathbf{B}, \mathbf{C}) < d(\mathbf{A}, \mathbf{C}) = d(\mathbf{A}, \mathbf{B})$. — This is true because the angle – sum property in Euclidean geometry implies that $|\angle \mathbf{ABC}| > 60°$ if $|\angle \mathbf{BAC}| < 60°$. However, the companion result does not hold in hyperbolic geometry. In fact, under these conditions for a fixed value of $|\angle \mathbf{BAC}|$ it is possible to construct triangles in a hyperbolic plane for which $|\angle \mathbf{ABC}| = |\angle \mathbf{ACB}|$ is arbitrarily small.